1 Problem

The expression for the $f_t$-value for unique first forbidden $\beta^-$ decay (derived in the lectures) is

$$f_{1u}^{1/2} = \frac{12\kappa}{g_\lambda^2 M_{1u}^2}$$

(1)

where $\kappa = 6147$ s, $g_\lambda = 1.25$, and

$$M_{1u} = \frac{2}{\sqrt{12\pi}} M_3 = \frac{2}{\sqrt{12\pi}} m_e c^2 R \frac{M^{(A)}}{\hbar c}$$

with $j_i = 2$ in our case, since we look at the transition $^{16}N(2^-) \rightarrow ^{16}O(0^+)$.

First of all, we need to evaluate the one-body transition density. Since $^{16}O$ is a doubly-closed shell nucleus, we use this as the Hartree-Fock vacuum, that is $|\text{HF}\rangle = |^{16}O(0^+)\rangle$. We find

$$\langle \psi_F | [c_\alpha^\dagger \bar{c}_\beta]_2 | \psi_I \rangle = (-1)^{m_I - j_f} \left( \begin{array}{cc} \frac{j_f}{-m_f} & \frac{j_i}{\mu} & \frac{2}{m_i} \end{array} \right)^{-1} \langle j_f m_f | [c_\alpha^\dagger \bar{c}_\beta]_2 | j_i m_i \rangle$$

$$= (-1)^{m_I - j_f} \left( \begin{array}{cc} \frac{j_f}{-m_f} & \frac{j_i}{\mu} & \frac{2}{m_i} \end{array} \right)^{-1} \sum_{m_\alpha m_\beta} (j_\alpha m_\alpha j_\beta m_\beta | 2\mu \rangle \langle j_f m_f | c_\alpha^\dagger \bar{c}_\beta | j_i m_i \rangle).$$

The states $\langle j_f m_f \rangle$ and $| j_i m_i \rangle$ are defined with respect to the HF vacuum as

$$\langle j_f m_f \rangle = \langle \text{HF} | \delta_{j_f 0} \delta_{m_f 0} \rangle$$

$$| j_i m_i \rangle = [c_\nu^\dagger h_\pi^\dagger]_{j_i m_i} | \text{HF} \rangle,$$

where $\nu$ denotes a neutron, $\pi$ a proton, $j_f = 0$, and $j_i = 2$. With these definitions we find

$$\langle j_f m_f | c_\alpha^\dagger \bar{c}_\beta | j_i m_i \rangle = \delta_{j_f 0} \delta_{m_f 0} \langle \text{HF} | c_\alpha^\dagger \bar{c}_\beta | c_\nu^\dagger h_\pi^\dagger \rangle_{j_i m_i} | \text{HF} \rangle$$

$$= \delta_{j_f 0} \delta_{m_f 0} \sum_{m_\alpha m_\beta} (j_\alpha m_\alpha j_\beta m_\beta | j_i m_i \rangle \langle \text{HF} | c_\alpha^\dagger \bar{c}_\beta c_\nu^\dagger h_\pi^\dagger | \text{HF} \rangle$$

$$= \delta_{j_f 0} \delta_{m_f 0} \sum_{m_\alpha m_\beta} (-1)^{1+j_\alpha+m_\alpha+j_\beta+m_\beta} (j_\nu m_\nu j_\pi m_\pi | j_i m_i \rangle$$

$$\langle \text{HF} | h_{a,-m_\alpha} c_{b,-m_\beta} c_{\nu,m_\nu} h_{\pi,m_\pi} | \text{HF} \rangle$$

$$= \delta_{j_f 0} \delta_{m_f 0} \sum_{m_\nu m_\pi} (-1)^{1+j_\alpha+m_\alpha+j_\beta+m_\beta} \delta_{a \nu} \delta_{m_\alpha,-m_\alpha} \delta_{b \nu} \delta_{m_\beta,-m_\beta} (j_\nu m_\nu j_\pi m_\pi | j_i m_i \rangle$$

$$= \delta_{j_f 0} \delta_{m_f 0} \delta_{a \nu} \delta_{b \nu} (-1)^{1+j_\alpha+m_\alpha+j_\beta+m_\beta} (j_b - m_\beta j_a - m_\alpha | j_i m_i \rangle,$$
where we have used Wick's theorem, the definition of $\tilde{c}$ and the particle-hole transformation to evaluate the matrix element. Next, we carry out the summations over $m_\alpha$ and $m_\beta$:

$$
\sum_{m_\alpha m_\beta} \left( j_a m_\alpha j_b m_\beta |2\mu\right) \delta_{j_f 0} \delta_{m_f 0} \delta_{\alpha \delta_{bw}} (-1)^{j_a + j_b + \mu} \\
(j_b - m_\beta j_a - m_\alpha |j_i m_i) \\
= \sum_{m_\alpha m_\beta} \left( j_a m_\alpha j_b m_\beta |2\mu\right) \delta_{j_f 0} \delta_{m_f 0} \delta_{\alpha \delta_{bw}} (-1)^{j_a + j_b + \mu} \\
= \sum_{m_\alpha m_\beta} \left( j_a m_\alpha j_b m_\beta |j_i - m_i\right) \\
= (-1)^{1+3j_a+3j_b+\mu} \delta_{j_f 0} \delta_{m_f 0} \delta_{\alpha \delta_{bw}}.
$$

Here we have used the fact that $m_\alpha + m_\beta = \mu$ for the Clebsh-Gordan coefficient $\left( j_a m_\alpha j_b m_\beta |2\mu\right)$ to be nonzero, $j_i = 2$ and the completeness relation for the Clebsh-Gordan coefficients. The one-body transition density is thus given by (since $j_f = 0 = m_f$ the phase in front of the $3j$ symbol drops out)

$$
\langle \psi_F || \left[ c_\alpha^\dagger \tilde{c}_b \right]_2 || \psi_I \rangle = \begin{pmatrix} 0 & 2 & 2 \\
0 & \mu & -\mu \end{pmatrix}^{-1} \delta_{j_f 2} \delta_{\mu, -m_i} \delta_{j_f 0} \delta_{m_f 0} \delta_{\alpha \delta_{bw}} \\
(-1)^{1+3j_a+3j_b+\mu} \\
= \sqrt{5} \delta_{j_f 2} \delta_{\mu, -m_i} \delta_{j_f 0} \delta_{m_f 0} \delta_{\alpha \delta_{bw}} (-1)^{1+3j_a+3j_b+\mu+2-\mu} \\
= (-1)^{1+3j_a+3j_b} \sqrt{5} \delta_{j_f 2} \delta_{\mu, -m_i} \delta_{j_f 0} \delta_{m_f 0} \delta_{\alpha \delta_{bw}},
$$

since the $3j$ symbol has the value

$$
\begin{pmatrix} 0 & 2 & 2 \\
0 & \mu & -\mu \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\
\mu & -\mu & 0 \end{pmatrix} = (-1)^{2-\mu} \frac{1}{\sqrt{5}}.
$$

We then find

$$
\sum_{ab} m_{21}^{(A)} (ab) \langle \psi_F || \left[ c_\alpha^\dagger \tilde{c}_b \right]_2 || \psi_I \rangle = m_{21}^{(A)} (\pi \nu) \sqrt{5} (-1)^{1+3j_a+3j_b}
$$

where $\pi$ denotes a proton hole in $0p_{1/2}$ and $\nu$ a neutron in $0d_{5/2}$. The relevant matrix element $m_{21}^{(A)} (\pi \nu)$ was evaluated during the course (exercise 29) and was found to have the value $-2b/\sqrt{5}=R/\sqrt{6}/5$ where $b$ is the oscillator parameter, and $R$ is the nuclear radius. By putting this into Equation (2) we finally have

$$
M^{(A)}(21) = -\frac{4\sqrt{\frac{6\pi}{5}b}}{R}.
$$
The dimensionless matrix element $M_{1u}$ is therefore

$$M_{1u} = \frac{1}{\sqrt{4\pi}} \frac{2 m_e e^2 R}{\hbar c} \left( -4\sqrt{\frac{6\pi}{5}} \frac{b}{R} \right)$$

$$= -\frac{8}{\sqrt{10}} m_e e^2 \frac{b R}{R \hbar c},$$

and the $ft$ value

$$f_{1u}t_{1/2} = \frac{12\kappa}{g_A^2 M_{1u}^2} = \frac{15}{8} \frac{\kappa}{g_A^2 \left( \frac{b R}{R \lambda_C^2} \right)^2},$$

where $\lambda_C^2 = \hbar/m_e c$ is the electron Compton wavelength. Thus we see that the $ft$-value for the forbidden case is larger than allowed $ft$ values by roughly a factor which goes like the square of the ratio of the electron Compton wavelength to the nuclear radius. The oscillator parameter $b$ is calculated using

$$b = \frac{\hbar c}{\sqrt{M_N c^2 \cdot \hbar \omega}}$$

where $\hbar \omega$ is estimated from the Blomquist-Molinari formula

$$\hbar \omega = \left( 45 A^{-1/3} - 25 A^{-2/3} \right) \text{MeV}.$$  

With $A = 16$ we get $\hbar \omega \approx 13.921$ MeV which gives $b = 1.722$ fm. Using $g_A = 1.25$ for the axial coupling constant we obtain the log($ft$) value

$$\log(f_{1u}t_{1/2}) \approx 8.569.$$  

In Exercise 30 we showed that in the Primakoff-Rosen (PR) approximation

$$f_{1u} \approx \frac{F_0^{PR}(Z_f)}{30} \left( \frac{4}{7} E_0^7 - E_0^5 - 10 E_0^4 + 30 E_0^3 - 32 E_0^2 + 15 E_0 - \frac{18}{7} \right),$$

where $E_0 = \Delta E/m_e c^2$, $\Delta E$ is the nuclear mass difference, $Z_f$ is the charge of the final nucleus, and

$$F_0^{PR}(Z_f) = \frac{2\pi \alpha Z_f}{1 - e^{-2\pi \alpha Z_f}},$$

where $\alpha \approx 1/137$ is the fine structure constant. In our case $Z_f = 8$, and so $F_0^{PR}(8) \approx 1.1946$. From ‘Table of Isotopes’ we find the $Q$-value for this transition to be

$$Q = 10419 \text{ keV}.$$
and then
\[ E_0 = \frac{Q}{m_e c^2} + 1 \approx 21.389. \]

This gives \( f_{1u} \approx 4.635 \cdot 10^7 \), and finally
\[ t_{1/2} = 10^{\log(f_{1u}) - \log(f_{1u})} \approx 8.0 \text{ s}. \]

In ‘Table of Isotopes’ we also find the experimental \( \log(ft) \)-value and the total half life for this transition:
\[
\log(ft)^{\text{exp}} = 9.1 \\
T_{1/2}^{\text{exp}} = 7.13 \text{ s}.
\]

From the total half life and the branching ratio to the ground state \( x = 28\% \), we can calculate the partial half life for this transition:
\[ t_{1/2} = \frac{T_{1/2}}{x} = 25.5 \text{ s} \]

As one could already see from the \( \log(ft) \)-value, the partial half-life is underestimated by about a factor of 3.
2 Problem

The given expression of \( \Gamma^{EC} \) is dependent on the following matrix elements:

\[
M_0(l) = g_V M_0^{(V)}(l) - g_A \frac{M_R^{(A)}(l)}{2 M c^2}
\]

\[
M(Ll) = g_V \frac{M_R^{(V)}(Ll)}{2 M c^2} - g_A M^{(A)}(Ll)
\]

Considering only allowed transitions, i.e. no recoil terms \( M_R \) and only s-waves with \( l = 0 \), reduces the number of possibly contributing matrix elements to the following ones:

\[
M_0(0) = g_V M_0^{(V)}(0)
\]

\[
M(L0) = -g_A M^{(A)}(L0)
\]

The matrix element \( M^{(A)}(L0) \) is given by the following expression:

\[
M^{(A)}(L0) = \sqrt{\frac{4 \pi}{2 J_I + 1}} \sum_{ab} m_{L0}^{(A)}(a b) \langle \psi_F | | c_a^{L} \phi^L_0 \rangle \psi_I
\]

The small matrix element \( m_{L0}^{(A)}(a b) \) was defined in the following way:

\[
m_{L0}^{(A)}(a b) = (-1)^{j_a + j_b + L} (i)^{j_a + j_b} \frac{1 + (-1)^{j_a + j_b}}{2} j_a j_b \left( \begin{array}{ccc} j_a & j_b & L \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{array} \right)
\]

\[
[A_0 L(a b) + B_0 L(a b)] R_{n_a n_b}^{(0)}
\]

The two functions \( A_0 L(a b) \) and \( B_0 L(a b) \) are given by the following expressions:

\[
A_0 L(a b) = \frac{j_a^2 + (-1)^{j_a + j_b + L} j_b^2}{\sqrt{2 L(L + 1)}} \left( \begin{array}{ccc} L & 1 & 0 \\ 1 & -1 & 0 \end{array} \right)
\]

\[
B_0 L(a b) = (-1)^{j_a + \frac{1}{2}} \left( \begin{array}{ccc} L & 1 & 0 \\ 0 & 0 & 0 \end{array} \right)
\]

From the properties of the 3j symbols, one can easily derive, that the only non-vanishing term for both functions, is that one with \( L = 1 \). Therefore, the number of possibly contributing matrix elements is further reduced to:

\[
M_0(0) = g_V M_0^{(V)}(0)
\]

\[
M(10) = -g_A M^{(A)}(10)
\]
Using the definitions of the Fermi $M_F$ and Gamov-Teller $M_{GT}$ matrix elements,

\[
M_F = \frac{1}{\sqrt{4\pi}} M_0^{(V)}(0)
\]

\[
M_{GT} = \frac{1}{\sqrt{4\pi}} M^{(A)}(10)
\]

one obtains finally:

\[
M_0(0) = \sqrt{4\pi} g_V M_F
\]

\[
M(10) = -\sqrt{4\pi} g_A M_{GT}
\]

Since only these two matrix elements can contribute to the shape function $S_0$ of the allowed transitions, one can easily see, that $B_2(\ell\ell')$ and $B_3(\ell\ell'')$ are vanishing, and $B_1(\ell)$ reduces to the following expression:

\[
B_1(\ell) = B_1(0) = |M_0(0)|^2 + |M(10)|^2 = 4\pi (g_V^2 M_F^2 + g_A^2 M_{GT}^2)
\]

The shape function $S_0$ of the allowed transitions itself, is then given by the following equation:

\[
S_0 = B_1(0) = 4\pi (g_V^2 M_F^2 + g_A^2 M_{GT}^2)
\]

Inserting this equation in the expression for the transition rate $\Gamma^{(EC)}$ yields:

\[
\Gamma^{(EC)} = \frac{G_F^2}{c^3 h^4 \pi^2} \left( \frac{Z_i}{a_0} \right)^3 (g_V^2 M_F^2 + g_A^2 M_{GT}^2)(E_e + \Delta E)^2
\]

Now, it is convenient to express Bohrs radius $a_0$, using the fine-structure constant $\alpha$, i.e. $a_0 = \frac{\hbar}{\epsilon m_e \alpha}$, which results in:

\[
\Gamma^{(EC)} = \frac{G_F^2 \epsilon^3 m_e^3 \alpha^3}{c^3 h^4 \pi^2} Z_i^3 (g_V^2 M_F^2 + g_A^2 M_{GT}^2)(E_e + \Delta E)^2
\]

Written in another form:

\[
\Gamma^{(EC)} = \frac{G_F^2 \epsilon^3 m_e^3 \alpha^3}{c^3 h^4 \pi^2} Z_i^3 (g_V^2 M_F^2 + g_A^2 M_{GT}^2) \left( \frac{E_e + \Delta E}{m_e c^2} \right)^2
\]

Introducing the universal constant $\kappa = \frac{2\hbar^2 \pi^2 \ln 2}{G_F^2 m_e^2 c^2}$ yields:

\[
\Gamma^{(EC)} = \frac{2\pi \ln 2 (\alpha Z_i)^3}{\kappa} (g_V^2 M_F^2 + g_A^2 M_{GT}^2) \left( \frac{E_e + \Delta E}{m_e c^2} \right)^2
\]
Since $E_e$ is given by $E_e = m_e c^2 \left(1 - \frac{1}{2} (\alpha Z_i)^2\right)$ and by defining $E_0 = \frac{\Delta E}{m_e c^2}$ one can obtain the following expression:

$$\Gamma^{(EC)} = \frac{2\pi \ln 2 (\alpha Z_i)^3}{\kappa} \left( g_V^2 M_F^2 + g_A^2 M_{GT}^2 \right) \left(1 - \frac{1}{2} (\alpha Z_i)^2 + E_0\right)^2$$

Further, one can define $f^{(EC)}_0 = 2\pi (\alpha Z_i)^3 \left(1 - \frac{1}{2} (\alpha Z_i)^2 + E_0\right)^2$, yielding:

$$\Gamma^{(EC)} = \frac{\ln 2}{\kappa} \left( g_V^2 M_F^2 + g_A^2 M_{GT}^2 \right) f^{(EC)}_0$$

The half-life can be calculated by $t^{(EC)}_\frac{1}{2} = \frac{\ln 2}{\Gamma^{(EC)} f^{(EC)}_0}$, which leads to the final result:

$$f^{(EC)}_0 t^{(EC)}_\frac{1}{2} = \frac{\kappa}{g_V^2 M_F^2 + g_A^2 M_{GT}^2}$$
3 Problem

The expressions

\[ S_0 = \sum_l \left( \frac{E_v R}{\hbar c} \right)^{2l} \frac{B_1(l)}{[(2l + 1)!!]^2} - 2 \sum_{l, L = l \pm 1} \left( \frac{E_v R}{\hbar c} \right)^{L+l} \frac{B_2(ll)}{(2l + 1)!!(2L + 1)!!} + 4 \sum_{l > l'} \left( \frac{E_v R}{\hbar c} \right)^{l+l'} \frac{B_3(ll')}{(2l + 1)!!(2l' + 1)!!} \]

with

\[ B_1(l) = |M_0(l)|^2 + \sum_{L = l, l \pm 1} |M(LL)|^2 \]

\[ B_2(ll) = i \left( \begin{array}{ccc} l & L & 1 \\ 0 & 0 & 0 \end{array} \right) M_0(L) M(LL), \quad L = l \pm 1 \]

and

\[ B_3(ll') = \sum_{L = l, l \pm 1} i^{ll'} \frac{1 + (-1)^l l + 1}{2} \left( \begin{array}{ccc} l & 1 & L \\ 0 & 0 & -1 \end{array} \right) \left( \begin{array}{ccc} l' & 1 & L \\ 0 & 0 & -1 \end{array} \right) M(LL) M(LL') \]

are to be used to derive an explicit expression for the shape function of the first forbidden electron capture, \( S_0^{(EC)}(ff) \), in the form

\[ S_0^{(EC)}(ff) = A + \frac{2}{3} BW_0 + \frac{1}{9} CW_0^2, \quad W_0 = 1 - \frac{1}{2}(\alpha Z_i)^2 + E_0 \]

Energy conservation in electron capture gives \( E_v = \Delta E + E_e \). The nuclear energy difference is \( \Delta E = E_I - E_F \). In addition we have the relations

\[ E_0 = \frac{\Delta E}{m_e c^2}, \quad \varepsilon = \frac{E_e}{m_e c^2} \]

\[ E_e = m_e c^2 - b \quad b = \frac{1}{2} m_e c^2 (\alpha Z_i)^2 \]

where \( b \) is the electron binding energy. This leads to

\[ W_0 = 1 - \frac{1}{2}(\alpha Z_i)^2 + E_0 = \varepsilon + E_0 = \frac{E_v}{m_e c^2}. \quad (3) \]

The factor \( B_1(l) \) depends on the following matrix elements:

\[ M_0(l) = g_V M_0^{(V)}(l) - g_A \frac{M_R^{(A)}(l)}{2 M c^2} \quad (4) \]

\[ M(LL) = g_V \frac{M_R^{(V)}(LL)}{2 M c^2} - g_A M^{(A)}(LL) \quad (5) \]
In first forbidden transitions both s- and p- waves (l=0, 1) contribute. Because of parity restrictions the number of possibly contributing matrix elements reduces to the following ones:

\[ M_0(0) = \frac{-g_A}{2M_Nc^2} \quad M(L) = \frac{g_V}{2M_Nc^2} \quad M_0(1) = g_V \quad M(L) = -g_A \frac{M(A)(1)}{2M_Nc^2} \]

The recoil term \( M_0^2(0)/2M_Nc^2 \) is defined as the dimensionless nuclear matrix element \( M_6 \). The matrix element \( M_0^2(L) \) is given by the expression:

\[ M_0^2(L) = \frac{4\pi}{2J_I + 1} \sum_{ab} m_{L0}^{(VR)}(a \ b) (\psi_F||[c^\dagger \tilde{c}]_L||\psi_I) \]

The small matrix element \( m_{L0}^{(VR)}(a \ b) \) is defined as:

\[ m_{L0}^{(VR)}(a \ b) = (-1)^{j_a + j_b + L} \frac{1}{2} \left( \begin{array}{c} j_a \ j_b \ L \\ \frac{1}{2} \ -\frac{1}{2} \ 0 \end{array} \right) \quad [A_{0L}(a \ b) \left( R_{11}^{(+)} + R_{11}^{(-)} + R_{12}^{(+)} - R_{12}^{(-)} \right) + B_{0L}(a \ b) \left( R_{22}^{(+)} + R_{22}^{(-)} - R_{21}^{(+)} - R_{21}^{(-)} \right)] \]

The two functions \( A_{0L}(a \ b) \) and \( B_{0L}(a \ b) \) are given by the expressions:

\[ A_{0L}(a \ b) = \frac{j_a^2 \ - \ 1}{\sqrt{2L(L+1)}} \left( \begin{array}{cc} L & 0 \\ 1 & -1 \end{array} \right) \]

\[ B_{0L}(a \ b) = (-1)^{j_a + \frac{1}{2}} \left( \begin{array}{c} L \ 0 \\ 1 \ 0 \end{array} \right) \]

From the summation in \( B_1 \) the two possible values for \( L \) are \( L = 0, 1 \) (\( L = -1 \) is impossible), but from the properties of the 3j symbols, the only non-vanishing term for both functions, is that one with \( L = 1 \). Therefore, the factor \( M_0^2(L) / 2M_Nc^2 \) is reduced to \( M_0^2(1) / 2M_Nc^2 \) which is defined as the dimensionless nuclear matrix element \( M_6 \).

The function \( M_0^2(1) \) is defined as \( \left( \hbar c / R \right) (M_4 / m_e c^2) \). The summation in \( B_0 \) gives three different \( M_0^2(1) \) terms with \( L = 0, 1, 2 \):

\[ M_0^2(1) = \frac{4\pi}{2J_I + 1} \sum_{ab} m_{L1}^{(A)}(a \ b) (\psi_F||[c^\dagger \tilde{c}]_L||\psi_I) \]

The small matrix element \( m_{L1}^{(A)}(a \ b) \) is defined as:

\[ m_{L1}^{(A)}(a \ b) = (-1)^{j_a + j_b + L} \frac{1}{2} \left( \begin{array}{c} j_a \ j_b \ L \\ \frac{1}{2} \ -\frac{1}{2} \ 0 \end{array} \right) \quad [A_{1L}(a \ b) + B_{1L}(a \ b)] \]

\[ R_{11}^{(1)} \]

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where

\[
A_{1L}(a b) = \frac{j_a^2 + (-1)^{j_a+j_b+L} j_b^2}{\sqrt{2}L(L+1)} \begin{pmatrix} L & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}
\]

\[
B_{1L}(a b) = (-1)^{1+j_a+\frac{L}{2}} \begin{pmatrix} L & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}
\]

The function \(A_{1L}(a b)\) restricts the possible \(L\) values to \(L = 1, 2\), but \(B_{1L}(a b)\) will also be defined for \(L = 0\). This means that all the three different \(L\) values contribute to \(B_1(1)\). The factors \(M^A(L)\), \(L = 0, 1, 2\) are defined as \((hc/R)(\mathcal{M}_i/m_c^2)\) for \(i = 1, 2, 3\), respectively. To summarize, we have now obtained for \(B_1(0)\) and \(B_1(1)\):

\[
B_1(0) = [M_0(0)]^2 + [M(10)]^2
\]

\[
= g_A^2 \left[ \frac{M_0^{(A)}(0)}{2M_Nc^2} \right]^2 + g_V^2 \left[ \frac{M_R^{(V)}(1)}{2M_Nc^2} \right]^2
\]

\[
= g_A^2 M_5^2 + g_V^2 M_6^2 \tag{6}
\]

\[
B_1(1) = [M_0(1)]^2 + \sum_{L=0}^2 [M(L1)]^2
\]

\[
= g_V^2 \left[ M_0^{(V)}(1) \right]^2 + g_A^2 \sum_{L=0}^2 [M^{(A)}(L1)]^2
\]

\[
= g_V^2 \left( \frac{hc}{R m_c^2} \right)^2 + g_A^2 \sum_{i=1}^3 \left( \frac{hc}{R m_c^2} \right)^2 \tag{7}
\]

The only possibility of \(B_2(L0)\) is \(L = 1\), since \(L = l \pm 1 = \pm 1\):

\[
B_2(10) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} M_0(1)M(10)
\]

\[
= -\frac{1}{\sqrt{3}} \left[ g_V M_0^{(V)}(1) \right] \begin{pmatrix} M_R^{(V)}(10) \\ \frac{M_R^{(V)}(10)}{2M_Nc^2} \end{pmatrix}
\]

\[
= -\frac{1}{\sqrt{3}} \left( g_V \frac{hc}{R m_c^2} M_4 \right) (g_V M_6) \tag{8}
\]

For \(B_2(L1)\) there are the two possible \(L\) values \(L = 0, 2\):

\[
B_2(01) = \sqrt{3} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} M_0(0)M(01)
\]

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\[
\begin{align*}
= \sqrt{3} \left( -\frac{1}{\sqrt{3}} \right) \left[ -\frac{g_A M_R^{(A)}(0)}{2 M_N c^2} \right] - g_A M^{(A)}(0, 1) \\
= - (g_A M_5) \left( g_A \frac{\hbar c M_1}{R m_N c^2} \right)
\end{align*}
\]

(9)

Because of the parity restrictions (recall Equations (4) and (5)), \(B_2(21)\) depends on the matrix elements:

\[
M_6(2) = -g_A \frac{M_R^{(A)}(2)}{2 M c^2} \quad M(21) = g_V \frac{M_R^{(V)}(21)}{2 M c^2}
\]

but these recoil terms can be neglected for a d-wave (\(L = 2\)) because of the suppression by the nucleon mass.

For the factor \(B_3(l l')\), let us now study the factor \(B_3(l 0)\):

\[
B_3(l 0) = \sum_{L=1, l \pm 1 \atop L=0,1} i^{1+(-1)^{l+1}} \frac{1}{2} \begin{pmatrix} l & 1 & L \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & L \\ 0 & 1 & -1 \end{pmatrix} M(LI)M(L0)
\]

The second 3\(j\) symbol above restricts \(L\) to 1, and the factor \(\frac{1+(-1)^{l+1}}{2}\) restricts \(l\) to 1, when \(l' = 0\). This agrees well with the summation limit, \(l > l'\), of the last term of the shape function.

\[
B_3(l l') = B_3(10) = \sqrt{3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} M(11)M(10)
\]

\[
= \sqrt{3} \left( \frac{1}{\sqrt{6}} \right) \left( \frac{1}{\sqrt{3}} \right) \left( -g_A \frac{\hbar c M_2}{R m_N c^2} \right) (g_V M_6)
\]

\[
= - \frac{1}{\sqrt{6}} g_V g_A M_6 \frac{M_2}{m_N c^2} \frac{\hbar c}{R}
\]

(10)

Recalling Equations (6) to (10) we are finally ready to derive the shape function of the first-forbidden electron capture:

\[
S_0^{(EC)}(ff) = \sum_l \left( \frac{E_\nu R}{\hbar c} \right)^{2l} \frac{B_1(l)}{(2l + 1)!!^2}
\]

\[
-2 \sum_{l, L=0, \pm 1} \left( \frac{E_\nu R}{\hbar c} \right)^{L+l} \frac{B_2(Ll)}{(2l + 1)!!(2L + 1)!!}
\]

\[
+4 \sum_{l, l' > l} \left( \frac{E_\nu R}{\hbar c} \right)^{l+l'} \frac{B_3(l l')}{(2l + 1)!!(2l' + 1)!!}
\]

\[
= B_1(0) + \left( \frac{E_\nu R}{\hbar c} \right)^2 \frac{B_1(1)}{9} - 2 \frac{E_\nu R}{\hbar c} \left[ \frac{B_2(10)}{3} + \frac{B_2(01)}{3} \right] + 4 \frac{E_\nu R B_3(10)}{3}
\]

(11)
\[
\begin{align*}
&= g_A^2 M_5^2 + g_V^2 M_6^2 \\
&\quad + \frac{1}{9} E_\nu^2 \left\{ g_V^2 \left( \frac{M_4}{m_c c^2} \right)^2 + g_A^2 \left[ \left( \frac{M_1}{m_c c^2} \right)^2 + \left( \frac{M_2}{m_c c^2} \right)^2 + \left( \frac{M_3}{m_c c^2} \right)^2 \right] \right\} \\
&\quad + \frac{2}{3} E_\nu \left[ \frac{1}{\sqrt{3}} g_V^2 \frac{M_4}{m_c c^2} M_6 + g_A^2 M_5 \frac{M_1}{m_c c^2} \right] \\
&\quad - \frac{4}{3\sqrt{6}} E_\nu g_V g_A M_6 \frac{M_2}{m_c c^2} \\
&= g_A^2 M_5^2 + g_V^2 M_6^2 \\
&\quad + \frac{2}{3} \left( \frac{E_\nu}{m_c c^2} \right) \left( \frac{1}{\sqrt{3}} g_V^2 M_4 M_6 + g_A^2 M_1 M_5 - \frac{2}{3} g_V g_A M_6 M_2 \right) \\
&\quad + \frac{1}{9} \left( \frac{E_\nu}{m_c c^2} \right)^2 \left[ g_V^2 M_4^2 + g_A^2 \left( M_1^2 + M_2^2 + M_3^2 \right) \right] \\
&= A + \frac{2}{3} BW_0 + \frac{1}{9} CW_0^2
\end{align*}
\]

where
\[
\begin{align*}
A &= g_A^2 M_5^2 + g_V^2 M_6^2 \\
B &= \left( \frac{1}{\sqrt{3}} g_V^2 M_4 M_6 + g_A^2 M_1 M_5 - \frac{2}{3} g_A g_V M_2 M_6 \right) \\
C &= \left[ g_V^2 M_4^2 + g_A^2 \left( M_1^2 + M_2^2 + M_3^2 \right) \right]
\end{align*}
\]
4 Problem

Discussion of the transition: \( ^{41}\text{Ar}(7/2^-) \xrightarrow{\beta^-} ^{41}\text{K}(3/2^+) \).

The expression derived in the lectures for the \( ft \)-value for unique first forbidden \( \beta^- \) decay is

\[
f_{1u \ell \ell \ell} = \frac{12\kappa}{g_A^2 M_{1u}^2},
\]

where \( \kappa = 6147 \text{ s}, g_A = 1.25 \). The matrix element \( M_{1u} \) is given by

\[
M_{1u} = \frac{2}{\sqrt{12\pi}} m_e c^2 \frac{R}{\hbar c} M^{(A)}(21)
\]

\[
= \frac{2}{\sqrt{12\pi}} m_e c^2 \frac{R}{\hbar c} \sqrt{\frac{4\pi}{2j_i + 1}} \sum_{ab} m_{21}^{(A)}(ab)(\psi_f || c^4_b \bar{c}_b || \psi_i). 
\]

The reduced matrix element \( (\psi_f || c^4_b \bar{c}_b || \psi_i) \) can be expressed as

\[
(\psi_f || c^4_b \bar{c}_b || \psi_i) = (-1)^{m_f - j_f} \left( \begin{array}{ccc} j_f & \frac{2}{\mu} & j_i \\ -m_f & 0 & m_i \end{array} \right)^{-1} \frac{\langle j_f m_f || c^4_b \bar{c}_b || j_i m_i \rangle}{M}.
\]

In order to find the value of the matrix element \( M \), we need expressions for the final and initial wavefunctions \( \langle \psi_f \rangle = \langle j_f m_f \rangle \) and \( |\psi_i\rangle = |j_i m_i\rangle \). The matrix element will be evaluated using the magic number 20 as the Hartree Fock vacuum for both neutrons and protons. The nucleus \( ^{41}\text{Ar} \) consists of 18 protons and 23 neutrons, and the ground state may be expressed as a two-hole three-particle state \( \pi(d_{3/2})^{-2} \times \nu(f_{7/2})^2 \times \nu(f_{7/2}) \) where \( \nu \) denotes a neutron and \( \pi \) denotes a proton. Similarly, the nucleus \( ^{41}\text{K} \) with 19 protons and 22 neutrons may be expressed as \( \pi(d_{3/2})^{-1} \times \nu(f_{7/2})^2 \). The wavefunctions with respect to the HF vacuum in the occupation number representation will then be

\[
\langle j_f m_f \rangle = \frac{1}{\sqrt{2}} \langle \text{HF} | h_{\pi}(d_{3/2}) c^4_{\bar{b}}(f_{7/2}) c^4_{\bar{b}}(f_{7/2}) \rangle_0
\]

\[
|j_i m_i\rangle = \frac{1}{\sqrt{12}} c^4_{\bar{b}}(f_{7/2}) c^4_{\bar{b}}(f_{7/2}) c^4_{\bar{b}}(f_{7/2})_0 h_{\pi}^\dagger(d_{3/2}) h_{\pi}^\dagger(d_{3/2})_0 \langle \text{HF} \rangle.
\]

This yields for the matrix element \( M \)

\[
M = \frac{1}{2 \sqrt{6}} \langle \text{HF} | h_{j_f, m_f}(\pi) c^4_{\bar{b}}(f_{7/2}) c^4_{\bar{b}}(f_{7/2}) c^4_{\bar{b}}(f_{7/2})_0 h_{\pi}^\dagger(d_{3/2}) h_{\pi}^\dagger(d_{3/2})_0 \langle \text{HF} \rangle.
\]

\[
c^4_{\bar{b}}(f_{7/2})(\nu) c^4_{\bar{b}}(f_{7/2})(\nu) c^4_{\bar{b}}(f_{7/2})_0 h_{\pi}^\dagger(d_{3/2}) h_{\pi}^\dagger(d_{3/2})_0 \langle \text{HF} \rangle.
\]

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The $\tilde{c}_b$ operator removes a neutron from above the Fermi level and will be replaced with $(-1)^{j_a+m_{\rho}} c_{j_b,-m_{\rho}}^{(i)}(\nu)$. The $c_b^{(i)}$ operator creates a proton below the Fermi level, and will be replaced by $-\hat{h}_a(\pi) = (-1)^{1+j_a+m_{a}} h_{j_a,-m_{a}}^{(i)}$. The expression for $M$ is then

$$M = \frac{1}{2\sqrt{6}} \sum_{m_2,m_3,m_4,m_5,m_6,m_7} (-1)^{j_a+m_{a}+j_{b}+m_{b}}$$

$$\langle HF | h_{j_f,m_f}^{(i)}(\pi) c_{j_3,m_3}^{(i)}(\nu) c_{j_4,m_4}^{(i)}(\nu) h_{j_a,-m_{a}}^{(i)}(\pi) c_{j_b,-m_{b}}^{(i)}(\nu) c_{j_5,m_5}^{(i)}(\nu) h_{j_7,m_{7}}^{(i)}(\pi) h_{j_8,m_{8}}^{(i)}(\pi) | HF \rangle$$

The Clebsch-Gordan coefficients that couple to zero are particularly easy to handle, and the contractions are taken over the following expression:

$$M = \frac{1}{2\sqrt{6}} \sum_{m_2,m_3,m_4,m_5,m_6,m_7} (-1)^{j_a+m_{a}+j_{b}+m_{b}} \sqrt{(2j_2+1)(2j_5+1)(2j_7+1)}$$

$$\delta_{j_2,j_3} \delta_{m_2,-m_3} \delta_{j_3,j_4} \delta_{m_3,-m_4} \delta_{j_4,j_5} \delta_{m_4,-m_5} \delta_{j_5,j_6} \delta_{m_5,-m_6} \delta_{j_6,j_7} \delta_{m_6,-m_7} (j_{a}m_{a}j_{b}m_{b})^{2\mu}$$

$$\langle HF | h_{j_f,m_f}^{(i)} c_{j_3,m_3}^{(i)} c_{j_4,m_4}^{(i)} h_{j_a,-m_{a}}^{(i)} c_{j_b,-m_{b}}^{(i)} c_{j_5,m_5}^{(i)} h_{j_7,m_{7}}^{(i)} h_{j_8,m_{8}}^{(i)} | HF \rangle$$

Contractions which involve both $h^{(i)}$ and $c^{(i)}$ operators do not contribute, thus only 12 contractions are non-vanishing.

$$M = \frac{1}{2\sqrt{6}} \sum_{m_2,m_3,m_4,m_5,m_6,m_7} (-1)^{m_{a}+m_{b}-m_{2}-m_{5}} \sum_{m_2,m_3,m_4,m_5,m_6,m_7}$$

$$\delta_{j_2,j_3} \delta_{m_2,-m_3} \delta_{j_3,j_4} \delta_{m_3,-m_4} \delta_{j_4,j_5} \delta_{m_4,-m_5} \delta_{j_5,j_6} \delta_{m_5,-m_6} \delta_{j_6,j_7} \delta_{m_6,-m_7} (j_{a}m_{a}j_{b}m_{b})^{2\mu}$$

$$\langle HF | h_{j_f,m_f}^{(i)} c_{j_3,m_3}^{(i)} c_{j_4,m_4}^{(i)} h_{j_a,-m_{a}}^{(i)} c_{j_b,-m_{b}}^{(i)} c_{j_5,m_5}^{(i)} h_{j_7,m_{7}}^{(i)} h_{j_8,m_{8}}^{(i)} | HF \rangle$$

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Summing over \(m_2, m_5\) and \(m_7\) and recalling that the different \(j\) values depend on which shell is involved, i.e. \(j_f = j_7 = j_8 = j_4\) and \(j_2 = j_3 = j_1 = j_5 = j_6 = j_0\) results in

\[
M = \frac{1}{2\sqrt{6}} \delta_{a,f} \delta_{b,i} \frac{(-1)^{1+2j_f+3j_i}}{(2j_i + 1)\sqrt{(2j_f + 1)}} \sum_{m_3 m_\alpha m_\beta m_\delta m_\epsilon} (-1)^{m_\alpha + m_\beta + m_3 + m_6 + m_8} (j_f m_\alpha j_i m_\beta | 2\mu)
\]

\[
\cdot \left[ \delta_{m_f, m_8} \delta_{m_\alpha, -m_\beta} \delta_{m_3, m_\delta} \delta_{m_6, -m_\epsilon} \delta_{m_\beta, m_\delta} \\
+ \delta_{m_f, m_8} \delta_{-m_\alpha, m_\beta} \delta_{m_3, -m_\delta} \delta_{m_6, m_\epsilon} \delta_{-m_\beta, m_\delta} \\
- \delta_{m_f, m_8} \delta_{-m_\alpha, -m_\beta} \delta_{m_3, m_\delta} \delta_{m_6, -m_\epsilon} \delta_{m_\beta, -m_\delta} \\
- \delta_{m_f, m_8} \delta_{m_\alpha, m_\beta} \delta_{m_3, -m_\delta} \delta_{-m_6, m_\epsilon} \delta_{-m_\beta, -m_\delta} \\
- \delta_{m_f, m_8} \delta_{-m_\alpha, -m_\beta} \delta_{m_3, m_\delta} \delta_{-m_6, -m_\epsilon} \delta_{m_\beta, m_\delta} \\
- \delta_{m_f, m_8} \delta_{m_\alpha, m_\beta} \delta_{m_3, -m_\delta} \delta_{m_6, -m_\epsilon} \delta_{-m_\beta, -m_\delta} \\
+ \delta_{m_f, m_8} \delta_{-m_\alpha, -m_\beta} \delta_{m_3, m_\delta} \delta_{-m_6, m_\epsilon} \delta_{m_\beta, m_\delta} \\
+ \delta_{m_f, m_8} \delta_{m_\alpha, m_\beta} \delta_{m_3, -m_\delta} \delta_{m_6, -m_\epsilon} \delta_{-m_\beta, -m_\delta} \right].
\]

Summing over \(m_3, m_\alpha, m_\beta, m_6\) and \(m_8\) results in the following:

\[
M = \frac{1}{2\sqrt{6}} \delta_{a,f} \delta_{b,i} \frac{(-1)^{1+2j_f+3j_i}(j_f m_f j_i - m_\epsilon | 2\mu)}{(2j_i + 1)\sqrt{(2j_f + 1)}} \left[ (-1)^{2m_f - m_\epsilon} \left( \left\{ \sum_{m_6} (-1)^{2m_6} \right\} + 1 \right) + 1 - \left\{ \sum_{m_6} 1 \right\} - (-1)^{-2m_\epsilon} - (-1)^{2m_\epsilon} \right] \\
(-1)^{-m_\epsilon} \left( \left\{ - \sum_{m_6} (-1)^{2m_6} \right\} - 1 - 1 \right) + \left\{ \sum_{m_6} 1 \right\} + (-1)^{-2m_\epsilon} + (-1)^{2m_\epsilon} \right]
\]

Making use of the fact that \(m_\epsilon, m_f\) and \(m_6\) are all half integers, \((-1)^{\pm 2m_\epsilon}\), \((-1)^{2m_f}\) and \((-1)^{2m_6}\) give just the factor \((-1)\). Then the last expression may be simplified to:

\[
M = \frac{1}{2\sqrt{6}} \delta_{a,f} \delta_{b,i} \frac{(-1)^{1+2j_f+3j_i}}{(2j_i + 1)\sqrt{(2j_f + 1)}} (j_f m_f j_i - m_\epsilon | 2\mu) (-1)^{-m_\epsilon}
\]

\[
\left[ \left\{ \sum_{m_6} 1 \right\} - 1 - 1 \right] + \left\{ \sum_{m_6} 1 \right\} - 1 - 1
\]

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\[ \left\{ \sum_{m_i} 1 \right\} - 1 - 1 + \left\{ \sum_{m_i} 1 \right\} - 1 - 1 \]
\[ = \frac{1}{2 \sqrt{6}} \delta_{a,f} \delta_{b,i} \frac{(-1)^{1+j_f+3j_i}}{(2j_i + 1) \sqrt{(2j_f + 1)}} \]
\[ (j_f m_f j_i - m_i |2\mu|(-1)^{-m_i} [4(2j_i + 1) - 8] \]

The reduced matrix element \((\psi_f ||c_a^i \tilde{c}_b|| \psi_i)\) defined in Equation 13 is then
\[ (\psi_f ||c_a^i \tilde{c}_b|| \psi_i) = \left( j_f \begin{array}{cc} 2 \end{array} \begin{array}{cc} j_i \end{array} \frac{\delta_{a,f} \delta_{b,i}(-1)^{1+j_f+3j_i}}{(2j_i + 1) \sqrt{2j_f + 1}} \right) \]
\[ \frac{1}{2 \sqrt{6}} [4(2j_i + 1) - 8] \sqrt{5} (-1)^{j_f-j_i-\mu} \left( j_f \begin{array}{cc} j_i \end{array} \begin{array}{cc} m_f \end{array} \begin{array}{cc} m_i \end{array} \begin{array}{cc} -\mu \end{array} \right) . \]

Using the relation between the Clebsch-Gordan coefficients and 3\(j\)-symbols the last expression is equal to
\[ (\psi_f ||c_a^i \tilde{c}_b|| \psi_i) = \left( j_f \begin{array}{cc} 2 \end{array} \begin{array}{cc} j_i \end{array} \right) (-1)^{1+j_f+3j_i} \frac{\delta_{a,f} \delta_{b,i}}{(2j_i + 1) \sqrt{2j_f + 1}} \]
\[ \frac{1}{2 \sqrt{6}} [4(2j_i + 1) - 8] \sqrt{5} (-1)^{j_f-j_i-\mu} \left( j_f \begin{array}{cc} j_i \end{array} \begin{array}{cc} m_f \end{array} \begin{array}{cc} m_i \end{array} \begin{array}{cc} -\mu \end{array} \right) . \]

Using the symmetry properties of the 3\(j\) symbol and the relations \(j_f = 3/2, \ j_i = 7/2\) and \(m_f - m_i = \mu\), the reduced matrix element may be expressed as
\[ (\psi_f ||c_a^i \tilde{c}_b|| \psi_i) = -\frac{3}{4} \sqrt{\frac{5}{6}} \delta_{a,f} \delta_{b,i} . \]

The matrix element \(M^{(A)}(21)\) is then expressed in terms of the sum over \(a\) and \(b\) of the small matrix elements \(m_{21}^{(A)}(ab)\). In this case only the matrix element \(m_{21}^{(A)}(fi)\), where \(f\) and \(i\) denotes the final \(d_{3/2}\) and initial \(f_{7/2}\) states respectively, contributes. This matrix element was calculated in the exercises to be
\[ m_{21}^{(A)}(d_{3/2}f_{7/2}) = -\frac{b}{R} \left( \frac{12\sqrt{2}}{5} \right) , \]

where \(R\) is the radius of the nucleus and \(b\) is the oscillator parameter. The value of the matrix element \(M^{(A)}(21)\) defined in Equation 12 is then
\[ M^{(A)}(21) = \frac{3b}{R} \sqrt{\frac{3\pi}{10}}, \]

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which then gives

\[ M_{1u} = \frac{3b \, m_e c}{\sqrt{10\hbar}}. \]

The oscillator parameter \( b \) is calculated using

\[ b = \frac{197 \, \text{fm}}{\sqrt{940 \hbar \omega}}, \]

where \( \hbar \omega \) is estimated from the Blomquist-Molinari formula

\[ \hbar \omega = (45A^{-1/3} - 25A^{-2/3}). \]

With \( A = 41 \) we get \( \hbar \omega = 10.948 \), which gives \( b = 1.942 \) fm. Using \( g_A = 1.25 \) for the axial coupling constant we obtain the \( f_{1u}t_{1/2} \) value from Equation 11

\[ f_{1u}t_{1/2} = \frac{12\kappa}{g_A^2 M_{1u}^2} = 2.07 \cdot 10^9 \text{s}, \]

which gives the \( \log(f_{1u}t_{1/2}) \) value

\[ \log(f_{1u}t_{1/2}) = 9.32 \text{ (exp: 9.7)} \]

This is of reasonable size for a first forbidden transition. The value of \( f_{1u} \) is estimated using the Primakoff-Rosen approximation

\[ f_{1u} = \frac{P_0^{PR}(Z_f)}{30} \left( \frac{4}{7} E_0^7 - E_0^5 - 10 E_0^4 + 30 E_0^3 - 32 E_0^2 + 15 E_0 - \frac{18}{7} \right), \]

where \( E_0 = \Delta E/m_e c^2 \), \( \Delta E \) is the nuclear mass difference, \( Z_f \) is the charge of the final nucleus, and

\[ P_0^{PR}(Z_f) = \frac{2\pi \alpha Z_f}{1 - e^{-2\pi \alpha Z_f}}, \]

where \( \alpha \approx 1/137 \) is the fine structure constant. In the case of the transition \( ^{41}\text{Ar}(7/2^-) \rightarrow ^{41}\text{K}(3/2^+) \), \( Z_f = 19 \) which leads to \( P_0^{PR}(19) = 1.4982 \). The \( Q \)-value for this transition is taken from the Table of Isotopes to be

\[ Q_{\beta^-} = 2491.6 \text{ keV}. \]

For \( \beta^- \) decay, the value of \( E_0 \) is then given by

\[ E_0 = \frac{Q}{m_e c^2} + 1 = 5.876. \]
This leads to the estimated value of \( f_{1u} = 6210 \). The half life is then given as
\[
t_{1/2} = 10^{\log(f_{1u}t_{1/2})-\log(f_{1u})} \approx 92.8 \text{ h}.
\]
From the experimental total half life \( T_{1/2} = 109.34 \text{ m} \) and the branching ratio to the ground state \( x = 0.83\% \) one can calculate the partial half life:
\[
t_{1/2} = 220 \text{ h}
\]
The partial half life is underestimated by only a factor of 2.

**Discussion of the transition: \(^{41}\text{Ca}(7/2^-) \xrightarrow{EC} ^{41}\text{K}(3/2^+)\).**

The half life for electron capture can be expressed through the following expression taken from the lectures,
\[
\Gamma^{(EC)} = \frac{G_F^2}{4e^3\hbar^4\pi^2} \left( \frac{Z_i}{a_0} \right)^3 S_0^{EC} (E_e + \Delta E)^2,
\]
as
\[
t^{EC}_{1/2} = \frac{\ln 2}{\Gamma^{(EC)}},
\]
where \( Z_i \) is the proton number of the initial nucleus, \( \kappa = 6147\text{s} \) and \( \alpha \) is the fine structure constant. The transition \(^{41}\text{Ca}(7/2^-) \xrightarrow{EC} ^{41}\text{K}(3/2^+)\) is of the unique first forbidden type (UFF), which means that the matrix element \( \mathcal{M}_3 \) is the only contributing matrix element. The shape function for electron capture, calculated in Problem 3, is then simplified to
\[
S_0^{(EC)}(\text{UFF}) = \frac{1}{9} W_{\theta}^2 \mathcal{M}_3^2,
\]
where \( \mathcal{M}_3 \) is given by
\[
\mathcal{M}_3 = \frac{R m_e c}{\hbar} M^{(A)} (21) \quad (15)
\]
\[
= \frac{R m_e c}{\hbar} \sqrt{\frac{4\pi}{2j_i + 1}} \sum_{ab} m_{21}^{(A)} (ab) \langle \psi_f || [c_a^\dagger \tilde{c}_b]_2 || \psi_i \rangle.
\]
The reduced matrix element \( \langle \psi_f || [c_a^\dagger \tilde{c}_b]_2 || \psi_i \rangle \) is given by
\[
\langle \psi_f || [c_a^\dagger \tilde{c}_b]_2 || \psi_i \rangle = (-1)^{m_f-j_f} \left( \begin{array}{ccc}
\hat{j}_j & 2 & \hat{j}_i \\
-m_f & \mu & m_i
\end{array} \right)^{-1} \langle j_f m_f || [c_a^\dagger \tilde{c}_b]_2 || j_i m_i \rangle.
\]
As in the previous calculation, the Hartree Fock vacuum is chosen as 20 protons and 20 neutrons. The final nucleus is the same as in the last calculation,
while the initial nucleus, $^{41}$Ca, for this calculation consists of 20 protons and 21 neutrons. The initial and final wavefunctions are then

$$\langle j_f m_f | = \frac{1}{\sqrt{2}} \langle \text{HF} | h_{3/2} (d_{3/2}) | c^+_a (f_{7/2})^l c^+_b (f_{7/2})^l \rangle \right|_{\alpha}$$

$$| j_i m_i \rangle = c^+_a (f_{7/2}) | \text{HF} \rangle .$$

This yields for the matrix element $M$

$$M = \frac{1}{\sqrt{2}} \langle \text{HF} | h_{j_f, m_f} | c^+_a c^+_b c^+_c \rangle _{0} \langle c^+_a \delta_b | 2 \mu | c^+_a, m_i \rangle \right|_{\text{HF}} .$$

This time the $\delta_b$ operator removes a proton from below the Fermi level and it will be replaced by $h^{1}_{b, m_n}$. The $c^+_a$ operator creates a neutron above the Fermi level and it will be replaced by $c^+_a, m_a$. The expression for $M$ is then

$$M = \left\{ \begin{array}{l}
\frac{1}{\sqrt{2}} \sum_{m_2, m_3, m_a, m_\beta} (j_2 m_2 j_3 m_3 | 0 \rangle (j_a m_a j_b m_\beta | 2 \mu ) \\
\langle \text{HF} | h_{j_f, m_f} c^+_a c^+_b c^+_c h^{1}_{b, m_n} c^+_a, m_i \rangle \right|_{\text{HF}} .
\end{array} \right.$$

This time only two sets of contractions are different from zero. Taking the contractions and writing out the Clebsch-Gordan coefficient that couples to zero gives

$$M = \left\{ \begin{array}{l}
\frac{1}{\sqrt{2}} \sum_{m_2, m_3, m_a, m_\beta} (-1)^{j_2 - m_2} \sqrt{2j_2 + 1} \delta_{j_2, j_3} \delta_{m_2, - m_3} (j_a m_a j_b m_\beta | 2 \mu ) \\
\left[ \delta_{j_f, j_b} \delta_{m_f, m_\beta} \delta_{j_b, j_a} \delta_{m_3, m_a} \delta_{j_2, j_i} \delta_{m_2, m_i} \\
- \delta_{j_f, j_i} \delta_{m_f, m_\beta} \delta_{j_2, j_a} \delta_{m_3, m_a} \delta_{j_2, j_i} \delta_{m_2, m_a} \right] .
\end{array} \right.$$

From the Kronecker symbols $\delta_{j_m, j_n}$, one can see that $j_2 = j_3 = j_a = j_i$ and $j_b = j_f$, and the last equation takes the simpler form. Summing over $m_2$, $m_3$, $m_a$ and $m_\beta$ results in

$$M = \delta_{f, b} \delta_{i, a} \sqrt{2} \frac{(-1)^{j_i - m_i} \sqrt{2j_i + 1} (j_i - m_i j_f m_f | 2 \mu )}{\sqrt{2j_i + 1}} .$$

which can be expressed as

$$M = \delta_{f, b} \delta_{i, a} \sqrt{10} \frac{(-1)^{j_i - m_i + 2j_f - \mu} \sqrt{2j_i + 1}}{\sqrt{2j_i + 1}} \left( \begin{array}{c}
j_f \\
- m_f \mu \\
- m_i \end{array} \right),$$

$$
\begin{pmatrix}
\frac{j_f}{2} & j_i \\
- m_f & \mu \\
- m_i 
\end{pmatrix}
\right),
$$
when the relations between the Clebsch-Gordan coefficients and the symmetry properties of the $3j$ symbol are used. Noticing that $m_f - m_i = \mu$. The reduced matrix element $(\psi_f || s_a^i s_b^i || \psi_i)\) takes the value

$$(\psi_f || s_a^i s_b^i || \psi_i) = \sqrt{\frac{10}{2j_i + 1}} (-1)^{j_i + j_f} \delta_{f,b}\delta_{i,a} \left( \begin{array}{ccc} j_f & 2 & j_i \\ -m_f & \mu & m_i \\ j_f & 2 & j_i \end{array} \right)$$

$$= \sqrt{\frac{10}{2j_i + 1}} (-1)^{j_i + j_f} \delta_{f,b}\delta_{i,a}.$$ 

As in the previous calculation, the sum over the small matrix elements is reduced to only one contribution:

$$m_{21}^{(A)}(r_7/2d_{3/2}) = \frac{b}{R} \left( \frac{12\sqrt{2}}{5} \right)$$

The dimensionless matrix element $M_3$, defined in Equation 15 may then be evaluated:

$$M_3 = \frac{R m_c}{\hbar} \sqrt{\frac{4\pi}{2j_i + 1}} \sum_{ab} m_{21}^{(A)}(ab)(\psi_f || s_a^i s_b^i || \psi_i)$$

$$= \sqrt{\frac{4\pi}{8} \frac{b m_c}{\hbar} \frac{12\sqrt{2}}{5} (-1)^{10}}$$

$$= -\frac{6b m_c}{\hbar} \sqrt{\pi} \left( \frac{1}{5} \right)$$

(16)

The decay width from Equation 14 may then be calculated:

$$\Gamma^{(EC)} = \frac{G_F^2}{4\epsilon^2\hbar^4\pi^3} \left( \frac{Z_i}{a_0} \right)^{3} S_0^{EC} (E_e + \Delta E)^2$$

$$= \frac{G_F^2}{4\epsilon^2\hbar^4\pi^3} \left( \frac{Z_i}{a_0} \right)^{3} \frac{1}{9} g_A^2 W_0^2 M_3^2 (E_e + \Delta E)^2$$

Replacing the Bohr radius and introducing the constant $\kappa$, as in Problem 2, gives the following for the decay width:

$$\Gamma^{(EC)} = \frac{\ln 2 (Z_i \alpha)^3}{2\kappa} \left( \frac{1}{(m_c e^2)^2} \right) (E_e + \Delta E)^2 g_A^2 W_0^2 \frac{9}{9} M_3^2$$

Noting that

$$\frac{1}{(m_c e^2)^2} (E_e + \Delta E)^2 = \left( \frac{E_e}{m_c e^2} + \frac{\Delta E}{m_c e^2} \right)^2$$

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\[
\frac{1}{2}(\alpha Z_i)^2 + E_0 \right)
\]
\[
W_0^2;
\]

and replacing \( M_3 \) with the value found in Equation 16 gives the half life:

\[
t_{1/2} = \frac{2 \kappa}{(\alpha Z_i)^3 g_A^2 W_0^4} \frac{9}{36 \pi} \left( \frac{\hbar}{b m_e c} \right)^2
\]

Introducing

\[
f_{1u} = 2 \pi (\alpha Z_i)^3 W_0^4
\]

one can write:

\[
f_{1u} t_{1/2} = 5 \kappa \left( \frac{\hbar}{g_A b m_e c} \right)^2
\]

The oscillator parameter is the same as in the previous calculation \( b = 1.942 \) fm. This gives the \( \log(f_{1u}t_{1/2}) \) value:

\[
\log(f_{1u}t_{1/2}) = 8.89 \quad (\exp: 10.5)
\]

\( W_0 \) is evaluated, using

\[
W_0 = \frac{Q_{EC}}{m_e c^2}.
\]

The \( Q \) value of this reaction is taken from the Table of Isotopes:

\[
Q_{EC} = 421.4 \text{ keV},
\]

which yields \( W_0 = 0.825 \) and \( f_{1u} = 9.04 \cdot 10^{-3} \). This yields for the half life:

\[
t_{1/2} = 2.72 \cdot 10^3 \text{ y} \quad (\exp: 1.03 \cdot 10^5 \text{ y})
\]

Here, the half life was underestimated by nearly a factor of 40. The EC-decay in this case shows no branching to excited states, therefore, the experimental total half life is assumed to be equal to the partial half life of the decay to the ground state in the daughter nucleus.
5 Problem

The $\beta^+$ transitions $^{18}\text{Ne} \rightarrow ^{18}\text{F} \rightarrow ^{18}\text{O}$ are allowed transitions with $\Delta I = 1$ and no parity changes. Therefore, their $f_t$-values can be expressed in terms of the Gamov-Teller matrix element, as the only contributing matrix element:

$$f_{0t} = \frac{\kappa}{g^2 M^2_{GT}}$$

The Gamov-Teller matrix element is given by:

$$M_{GT} = \frac{1}{\sqrt{4\pi}} M^{(A)}(10)$$

The axial matrix element is given by the following equation:

$$M^{(A)}(10) = \sqrt{\frac{4\pi}{2j_1 + 1}} \sum_{ab} m_{20}^{(A)}(ab) (\psi_f||[c_{a}^+ \bar{c}_{b}]||\psi_i)$$

The nuclear wave functions are now calculated within the s-d shell, assuming, that the shell model orbitals $0d_{5/2}$ and $1s_{1/2}$ are almost degenerate and far from the $0d_{3/2}$ orbital. This means, that the ground state admixture of the $0d_{5/2}$ orbital is assumed to be zero. Since the configuration $(d_{5/2})(s_{1/2})$ can couple neither to a $1^+$ nor to a $0^+$ state, the only two configurations contributing to the ground states of the involved nuclei are $(0d_{5/2})^2$ and $(1s_{1/2})^2$ as shown below:

$$^{18}\text{Ne} = \frac{\alpha_{10}}{\sqrt{2}} |(0d_{5/2})^{2\pi} > + \frac{\beta_{10}}{\sqrt{2}} |(1s_{1/2})^{2\pi} >$$
$$^{18}\text{F} = \alpha_{20} |(0d_{5/2})^{+\pi^\prime} > + \beta_{20} |(1s_{1/2})^{+\pi^\prime} >$$
$$^{18}\text{O} = \frac{\alpha_{30}}{\sqrt{2}} |(0d_{5/2})^{2\nu} > + \frac{\beta_{30}}{\sqrt{2}} |(1s_{1/2})^{2\nu} >$$

The coefficients $\alpha$ and $\beta$ have now to be determined by solving the Schrödinger equation. The two degenerate states of the original eigenwertproblem are now denoted as:

$$\psi_0 = |(0d_{5/2})^{2n} >$$
$$\psi_1 = |(1s_{1/2})^{2n} >$$

Adding a residual interaction to the original Hamiltonian in the form $\hat{H} = \hat{H}_0 + \hat{V}$ one has to diagonalize the following matrix:

$$H = \begin{pmatrix}
2\varepsilon(0d_{5/2}) + <\psi_0|\hat{V}|\psi_0> & <\psi_0|\hat{V}|\psi_1>\\
<\psi_1|\hat{V}|\psi_0> & 2\varepsilon(1s_{1/2}) + <\psi_1|\hat{V}|\psi_1>
\end{pmatrix}$$

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Written in short hand notation:

\[ H = \begin{pmatrix} 2\varepsilon_0 + V_{00} & V_{01} \\ V_{10} & 2\varepsilon_1 + V_{11} \end{pmatrix} \]

Since \( \hat{V} \) is a hermitian operator, one obtains \( V_{01} = V_{01}^* \). Assuming further on that the matrix elements are real, one even obtains \( V_{01} = V_{10}^* = V_{10} \). Since the shell model orbitals \((0d_\frac{3}{2})^2\) and \((1s_\frac{1}{2})^2\) are degenerated, one can write \( 2\varepsilon_0 = 2\varepsilon_1 \equiv \varepsilon \). At last, the diagonal interaction matrix elements are roughly independent of the orbitals, which yields to \( V_{00} = V_{11} \equiv V \). Introducing all these approximations, and using the notation \( \varepsilon + V = W \) one obtains finally:

\[ H = \begin{pmatrix} W & V_{01} \\ V_{01} & W \end{pmatrix} \]

The eigenvalues of this matrix are now given by:

\[ \lambda_{0/1} = W \mp V_{01} \]

The eigenvectors of the two states, can be calculated from the following equation:

\[ \begin{pmatrix} \pm V_{01} & V_{01} \\ V_{01} & \pm V_{01} \end{pmatrix} \begin{pmatrix} \alpha_{i0} \\ \beta_{i0} \end{pmatrix} = 0 \]

Exploiting the normalization condition \( \alpha_{i0}^2 + \beta_{i0}^2 = 1 \) one obtains finally:

\[ \alpha_{i0} = \frac{1}{\sqrt{2}} \quad \beta_{i0} = \mp \frac{1}{\sqrt{2}} \]

The nuclear wave functions can now be written explicitly, using \(^{16}\text{O}\) as the Hartree Fock vacuum state:

\[ ^{18}\text{Ne} = \frac{1}{2}[c_x^+(0d_\frac{3}{2})c_{\frac{3}{2}}^+(0d_\frac{3}{2})][0]_{\text{HF}} > + \frac{1}{2}[c_x^+(1s_\frac{1}{2})c_{\frac{1}{2}}^+(1s_\frac{1}{2})][0]_{\text{HF}} > \]

\[ ^{18}\text{F}(1,2) = \frac{1}{\sqrt{2}}[c_x^+(0d_\frac{3}{2})c_{\frac{3}{2}}^+(0d_\frac{3}{2})][1]_{\text{HF}} > + \frac{1}{\sqrt{2}}[c_x^+(1s_\frac{1}{2})c_{\frac{1}{2}}^+(1s_\frac{1}{2})][1]_{\text{HF}} > \]

\[ ^{18}\text{O} = \frac{1}{2}[c_x^+(0d_\frac{3}{2})c_{\frac{3}{2}}^+(0d_\frac{3}{2})][0]_{\text{HF}} > + \frac{1}{2}[c_x^+(1s_\frac{1}{2})c_{\frac{1}{2}}^+(1s_\frac{1}{2})][0]_{\text{HF}} > \]

In the next step, a general expression for the nuclear matrix elements will be derived. Starting from the reduced matrix element and applying the Wigner-Eckart theorem, one gets:

\[ < f || [c_a^d \delta_i]_1 || i > = (-1)^{m_f - j_f} \begin{pmatrix} j_f & 1 & j_i \\ -m_f & \mu & m_i \end{pmatrix}^{-1} < f || [c_a^d \delta_i]_{1\mu} || i > \quad (17) \]
The nuclear matrix element has now the general form:

\[
<f|c_d^\dagger \tilde{c}_b|_\mu \nu > = <HF|c_d^\dagger c_b^\dagger F_j |c_a^\dagger \tilde{c}_a|_\mu |c_d^\dagger F_j |m_4|HF>
\]

\[
\sum_{m_1 m_2 m_3 m_4} <HF|c_{j_1 m_1} c_{j_2 m_2} c_{\alpha m_3} c_{b-m_\beta} c_{j_3 m_3} c_{j_4 m_4}|HF>
\]

\[
(-1)^{j_2 + m_\beta} (j_1 m_1 j_2 m_2 |j_f m_f)(j_a m_\alpha j_\beta m_\beta |1 \mu)(j_3 m_3 j_4 m_4 |j_f m_f)
\]

The Hartree-Fock ground state expectation value can now be calculated using Wicks theorem. In general, four different possibilities of contracting the expression given above, are contributing:

\[
<HF|c_{j_2 m_2} c_{j_1 m_1} c_{\alpha m_3} c_{b-m_\beta} c_{j_3 m_3} c_{j_4 m_4}|HF> =
\]

\[
+ c_{j_2 m_2} c_{\alpha m_3} c_{j_1 m_1} c_{j_3 m_3} c_{j_4 m_4} - c_{j_2 m_2} c_{\alpha m_3} c_{j_3 m_3} c_{j_1 m_1} c_{j_4 m_4} - c_{j_2 m_2} c_{\alpha m_3} c_{j_1 m_1} c_{j_3 m_3} c_{j_4 m_4}
\]

\[
- c_{j_2 m_2} c_{\alpha m_3} c_{j_3 m_3} c_{j_1 m_1} c_{j_4 m_4} + c_{j_2 m_2} c_{\alpha m_3} c_{j_1 m_1} c_{j_4 m_4} c_{j_3 m_3}
\]

\[
= \delta_{j_2 a} \delta_{m_2 m_\alpha} \delta_{j_1 j_3} \delta_{m_1 m_3} \delta_{j_4 b} \delta_{m_\beta m_4} - \delta_{j_2 a} \delta_{m_2 m_\alpha} \delta_{j_1 j_3} \delta_{m_1 m_4} \delta_{j_4 b} \delta_{m_\beta m_3}
\]

\[
- \delta_{j_2 j_3} \delta_{m_2 m_3} \delta_{j_1 a} \delta_{m_1 m_\alpha} \delta_{j_4 b} \delta_{m_\beta m_4}
\]

\[
+ \delta_{j_2 j_3} \delta_{m_2 m_3} \delta_{j_1 a} \delta_{m_1 m_\alpha} \delta_{j_4 b} \delta_{m_\beta m_3}
\]

Summing over \(m_3, m_4\) and \(m_\alpha\) yields:

\[
<f|c_d^\dagger \tilde{c}_b|_\mu \nu > = \sum_{m_1 m_2 m_\beta} (-1)^{j_2 + m_\beta}
\]

\[
[(j_1 m_1 j_2 m_2 |j_f m_f)(j_2 m_2 j_3 m_\beta |1 \mu)(j_1 m_1 j_\beta - m_\beta j_1 |j_\beta m_\beta)
\]

\[
- (j_1 m_1 j_2 m_2 |j_f m_f)(j_2 m_2 j_3 m_\beta |1 \mu)(j_1 m_1 j_\beta - m_\beta j_1 |j_\beta m_\beta)
\]

\[
- (j_1 m_1 j_2 m_2 |j_f m_f)(j_1 m_1 j_3 m_\beta |1 \mu)(j_2 m_2 j_\beta - m_\beta j_2 |j_\beta m_\beta)
\]

\[
+ (j_1 m_1 j_2 m_2 |j_f m_f)(j_1 m_1 j_3 m_\beta |1 \mu)(j_2 m_2 j_\beta - m_\beta j_2 |j_\beta m_\beta)
\]

By using the following properties of the Clebsch Gordan coefficients:

\[
(j_\beta - m_\beta j_1 m_1 j_\beta) = (-1)^{j_1 + j_\beta - j_2} (j_1 m_1 j_\beta - m_\beta j_1 |j_\beta m_\beta)
\]

\[
(j_\beta - m_\beta j_2 m_2 j_\beta) = (-1)^{j_2 + j_\beta - j_1} (j_2 m_2 j_\beta - m_\beta j_2 |j_\beta m_\beta)
\]

one can rewrite the equation in a more compact form:

\[
<f|c_d^\dagger \tilde{c}_b|_\mu \nu > = \sum_{m_1 m_2 m_\beta} (-1)^{j_2 + m_\beta}
\]

\[
[(j_1 m_1 j_2 m_2 |j_f m_f)(j_2 m_2 j_3 m_\beta |1 \mu)(j_1 m_1 j_\beta - m_\beta j_1 |j_\beta m_\beta)
\]

\[
(\delta_{j_2 a} \delta_{j_1 j_3} \delta_{j_4 b} - (-1)^{j_1 + j_\beta - j_2} \delta_{j_2 a} \delta_{j_1 j_3} \delta_{j_4 b})
\]

\[
- (j_1 m_1 j_2 m_2 |j_f m_f)(j_1 m_1 j_3 m_\beta |1 \mu)(j_2 m_2 j_\beta - m_\beta j_2 |j_\beta m_\beta)
\]

\[
(\delta_{j_2 j_3} \delta_{j_1 a} \delta_{j_4 b} - (-1)^{j_2 + j_\beta - j_1} \delta_{j_2 j_3} \delta_{j_1 a} \delta_{j_4 b})
\]
Writing the Clebsch-Gordan coefficients in terms of $3j$-symbols

\[
(j_1 m_1 j_2 m_2 | j_3 m_3) = (-1)^{j_1 - j_3 + m_3} j_3 \begin{pmatrix}
  j_1 & j_2 & j_3 \\
  m_1 & m_2 & -m_3
\end{pmatrix}
\]

and using Equation (108.4) from the book (L. D. Landau, E. M. Lifschitz, Lehrbuch der theorischen Physik, Band III, Quantenmechanik, Akademie-Verlag Berlin 1979)

\[
\sum_{m_4 m_5 m_6} (-1)^{j_4 + j_5 + j_6 - m_4 - m_5 - m_6} 
\begin{pmatrix}
  j_1 & j_2 & j_3 \\
  m_1 & m_2 & m_3
\end{pmatrix}
\begin{pmatrix}
  j_4 & j_5 & j_6 \\
  m_4 & m_5 & -m_6
\end{pmatrix}
\begin{pmatrix}
  j_4 & j_5 & j_6 \\
  -m_4 & m_5 & m_6
\end{pmatrix}
\]

one obtains:

\[
<i | c_a^j \bar{c}_b | \mu \nu > =
\]

\[
\begin{pmatrix}
  1 & j_i & j_f \\
  \mu & m_i & -m_f
\end{pmatrix}
\begin{pmatrix}
  1 & j_i & j_f \\
  j_j & j_l & j_b
\end{pmatrix}
\begin{pmatrix}
  j_f j_i \sqrt{3} (-1)^{j_i + j_b + 1 + 2j_i + m_f} \\
  j_f j_i \sqrt{3} (-1)^{j_i + j_b + 1 + 2j_i + m_f}
\end{pmatrix}
\]

\[
\begin{pmatrix}
  (\delta_{j_r a} \delta_{j_1 a} \delta_{b j_4} - (1)^{j_1 + j_b - j_i} \delta_{j_2 a} \delta_{j_4 a} \delta_{b j_3}) \\
  (\delta_{j_r a} \delta_{j_1 a} \delta_{b j_4} - (1)^{j_1 + j_b - j_i} \delta_{j_2 a} \delta_{j_4 a} \delta_{b j_3})
\end{pmatrix}
\]

For the reduced matrix element, given in Equation (17) one obtains:

\[
<i | c_a^j \bar{c}_b | \mu = j_f j_i \sqrt{3}
\]

\[
\begin{pmatrix}
  1 & j_i & j_f \\
  j_j & j_l & j_b
\end{pmatrix}
\begin{pmatrix}
  1 & j_i & j_f \\
  j_j & j_l & j_b
\end{pmatrix}
\begin{pmatrix}
  (1)^{j_1 + j_b + 1 + 2j_i + m_f} (\delta_{j_2 a} \delta_{j_1 a} \delta_{b j_4} - (1)^{j_1 + j_b - j_i} \delta_{j_2 a} \delta_{j_4 a} \delta_{b j_3}) \\
  (1)^{2j_f - j_1 + j_b + 1} (\delta_{j_1 a} \delta_{b j_4} - (1)^{j_2 + j_b - j_i} \delta_{j_1 a} \delta_{b j_4})
\end{pmatrix}
\]

For all partial wave functions involved in this calculation, the conditions $j_1 = j_2$ and $j_3 = j_4$ hold. Due to the Kronecker symbols in the Equation above, this yields $j_1 = j_2 = j_3 = j_4 = j_a = j_b$ $\equiv j$, otherwise the reduced matrix element yields zero. Inserting these conditions, one obtains:

\[
<i | c_a^j \bar{c}_b | \mu > = j_f j_i \sqrt{3}
\]

\[
\begin{pmatrix}
  1 & j_i & j_f \\
  j_j & j_l & j_b
\end{pmatrix}
\begin{pmatrix}
  1 & j_i & j_f \\
  j_j & j_l & j_b
\end{pmatrix}
\begin{pmatrix}
  (1)^{2j_f + 1 + 2j_i + j_f} (\delta_{j_a a} \delta_{b j} - (1)^{2j_f - j_1} \delta_{j_a a} \delta_{b j}) \\
  (1)^{2j_f + 1} (\delta_{j_a a} \delta_{b j} - (1)^{2j_f - j_1} \delta_{j_a a} \delta_{b j})
\end{pmatrix}
\]

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Since $j$ is a half integer spin, $2j$ is an odd integer. The initial and final spins $j_i$ and $j_f$, respectively, are integers in this calculation, therefore $2j_i$ and $2j_f$ are even integers. Using this, one can write the reduced nuclear matrix element in a very compact form:

$$< f | [c^A_d \delta_b]_1 | i > = j_f j_i \sqrt{3} \begin{pmatrix} 1 & j_i & j_f \\ j & j & j \end{pmatrix} \delta_{j_a \delta_{b_j}} (-1)^{j_i} + (-1)^{j_f + j_i} + 1 + (-1)^{j_i}$$

It is worth to notice, that the result is independent of the order of $c_{j_1m_1} c_{j_2m_2}$ or $c_{j_2m_2} c_{j_1m_1}$ as one would expect. Inserting the expression for the reduced nuclear matrix element in the expression for the Gamov-Teller matrix element, one gets:

$$M_{GT} = \sqrt{3} j_f m_{10}^{(A)} (j, j) \begin{pmatrix} 1 & j_i & j_f \\ j & j & j \end{pmatrix} \left[ (-1)^{j_i} + (-1)^{j_f + j_i} + 1 + (-1)^{j_i} \right]$$

Now, one can proceed by evaluating the matrix elements for the special case of the $\beta$-decay chain $^{18}$Ne$\rightarrow^{18}$F$\rightarrow^{18}$O in the s-d shell. First, the matrix element with $|i > = |^{18}$Ne $>$ and $|f > = |^{18}$F $>$ will be evaluated. Since $c_2 = c_\nu$, in this case, only the first and second contraction can contribute, due to the fact, that the contraction of a neutron annihilation operator and a proton creation operator yields zero. In addition, one has to respect the condition $j_1 = j_2 = j_3 = j_4$, therefore one obtains the simple result:

$$M_{GT}^{(1)} = -\frac{3}{\sqrt{2}} m_{10}^{(A)} \begin{pmatrix} 5 & 5 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ \frac{5}{2} & \frac{5}{2} & \frac{5}{2} \end{pmatrix} \pm m_{10}^{(A)} \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \right)$$

Now, the matrix element with $|i > = |^{18}$F $>$ and $|f > = |^{18}$O $>$ will be evaluated. In this case $c_3 = c_-^{(1)}$, therefore, only the second and forth contraction can contribute.

$$M_{GT}^{(2)} = -\sqrt{3} \frac{2}{2} m_{10}^{(A)} \begin{pmatrix} 5 & 5 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ \frac{5}{2} & \frac{5}{2} & \frac{5}{2} \end{pmatrix} + m_{10}^{(A)} \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \right)$$

The $6j$ symbols can be determined, using tabulated values.

$$\begin{pmatrix} 1 & 0 & 1 \\ \frac{5}{2} & \frac{5}{2} & \frac{5}{2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ \frac{5}{2} & \frac{5}{2} & \frac{5}{2} \end{pmatrix} = \frac{1}{3\sqrt{2}}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \frac{1}{\sqrt{6}}$$

The single particle matrix element $m_{10}^{(A)}(0d_{\frac{3}{2}}, 0d_{\frac{1}{2}}) = \sqrt{\frac{14}{5}}$ was derived in the exercises. The other matrix element will be calculated, using its definition:

$$m_{10}^{(A)}(a, b) = (-1)^{j_a + j_b + 1} (i)^{l_a + l_b} \frac{1}{2} (-1)^{j_a + l_b} j_a j_b \left( \begin{array}{c} j_a \frac{1}{2} \\ j_b \frac{1}{2} \end{array} \right)$$

$$[A_{a1}(a, b) + B_{a1}(a, b)] R_{n_a, l_a, n_b, l_b}^{(0)}$$

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The two functions $A_{01}(a b)$ and $B_{01}(a b)$ are given by the following expressions:

$$
A_{01}(a b) = \frac{j_a^2 + (-1)^{j_a+j_b+1} \bar{j_b}^2}{2} \begin{pmatrix}
1 & 1 & 0 \\
1 & -1 & 0
\end{pmatrix} = \frac{j_a^2 + (-1)^{j_a+j_b+1} \bar{j_b}^2}{2} \frac{1}{\sqrt{3}}
$$

$$
B_{01}(a b) = (-1)^{j_a+\frac{1}{2}} \begin{pmatrix}
1 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix} = -(-1)^{j_a+\frac{1}{2}} \frac{1}{\sqrt{3}}
$$

Inserting $(1s_{\frac{7}{2}})$ for $a$ and $b$ respectively yields:

$$
m^{(4)}_{10}(1s_{\frac{7}{2}}1s_{\frac{7}{2}}) = 2 \left( \frac{1}{2} \frac{1}{2} \frac{1}{2} 0 \right) \left( \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} \right) R_{1010}^{(0)}
$$

$$
= \sqrt{2} R_{1010}^{(0)}
$$

The term $R_{1010}^{(0)}$ can be expressed with radial wave functions:

$$
R_{1010}^{(0)} = \int_0^{\infty} R_{10} R_{10} r^2 dr
$$

Since the radial wave functions are normalized, the integral above simply yields 1, and the single particle matrix element $m^{(4)}_{10}(1s_{\frac{7}{2}}1s_{\frac{7}{2}}) = \sqrt{2}$. Inserting these expressions, one obtains for the Gamov-Teller matrix elements:

$$
M_{GT}^{(1)} = -\frac{3}{\sqrt{2}} \left( \sqrt{\frac{14}{5}} \frac{1}{3\sqrt{2}} \pm \sqrt{\frac{14}{5}} \frac{1}{3\sqrt{2}} \right) = -\sqrt{\frac{7}{10}} \pm \sqrt{\frac{3}{2}}
$$

$$
M_{GT}^{(2)} = -\sqrt{\frac{3}{2}} \left( \sqrt{\frac{14}{5}} \frac{1}{3\sqrt{2}} \pm \sqrt{\frac{14}{5}} \frac{1}{3\sqrt{2}} \right) = -\sqrt{\frac{7}{30}} \pm \frac{1}{\sqrt{2}}
$$

Now it is possible to calculate log $ft$ values, using $\kappa = 6147$ s and $g_A = 1.25$.

$$
fo_{\frac{1}{2}}^{(18}\text{Ne} \rightarrow ^{18}\text{F}(1)) \approx 926 \text{ s} \Rightarrow \log ft \approx 2.97 \text{ (exp: 3.1)}
$$

$$
fo_{\frac{1}{2}}^{(18}\text{Ne} \rightarrow ^{18}\text{F}(2)) \approx 7.26 \text{ h} \Rightarrow \log ft \approx 4.42 \text{ (exp: 4.5)}
$$

$$
fo_{\frac{1}{2}}^{(18}\text{F} \rightarrow ^{18}\text{O}) \approx 46.3 \text{ m} \Rightarrow \log ft \approx 3.44 \text{ (exp: 3.6)}
$$

The phase-space factor in the Primakov-Rosen approximation is given by:

$$
fo(E_0) = \frac{1}{30} (E_0^5 - 10 E_0^2 + 15 E_0 - 6) F_0^{(PR)}
$$

The Primakov-Rosen function $F_0^{(PR)}$ in the case of $\beta^+$ decay is given by:

$$
F_0^{(PR)} = \frac{2\pi \alpha Z_f}{e^{2\pi \alpha Z_f} - 1}
$$
Neglecting the binding energy, one can write:

$$E_0 = \frac{Q_{\text{sc}}}{m_e c^2} - 1$$

Now, one can calculate the phase space factors:

\begin{align*}
^{18}\text{Ne} \rightarrow ^{18}\text{F}(1) : & \quad Q_{EC} = 4446 \text{ keV} : \quad f_0 \approx 716 \\
^{18}\text{Ne} \rightarrow ^{18}\text{F}(2) : & \quad Q_{EC} = 2745 \text{ keV} : \quad f_0 \approx 39.5 \\
^{18}\text{F} \rightarrow ^{18}\text{O} : & \quad Q_{EC} = 1655.5 \text{ keV} : \quad f_0 \approx 0.932
\end{align*}

Finally, it is possible to calculate the half lifes:

\begin{align*}
^{18}\text{Ne} \rightarrow ^{18}\text{F}(1) : & \quad t_{1/2} \approx 1.29 \text{ s} \\
^{18}\text{Ne} \rightarrow ^{18}\text{F}(2) : & \quad t_{1/2} \approx 11.0 \text{ m} \\
^{18}\text{F} \rightarrow ^{18}\text{O} : & \quad t_{1/2} \approx 49.7 \text{ m}
\end{align*}

From the experimental total half lifes $T_{1/2} = 1672$ ms and $T_{1/2} = 109.77$ m and the branching ratios to the ground states $x = 92.11\%$ $x = 0.188\%$ and $x = 100\%$, respectively, one can calculate the experimental partial half lifes of the transitions:

\begin{align*}
^{18}\text{Ne} \rightarrow ^{18}\text{F}(1) : & \quad t_{1/2} \approx 1.82 \text{ s} \\
^{18}\text{Ne} \rightarrow ^{18}\text{F}(2) : & \quad t_{1/2} \approx 14.8 \text{ m} \\
^{18}\text{F} \rightarrow ^{18}\text{O} : & \quad t_{1/2} \approx 109.77 \text{ m}
\end{align*}

The log $ft$ values are in very good agreement with the experiment, whereas the partial half lifes are generally underestimated by 25-100%.