Building light nuclei from neutrons, protons, and pions

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In these lectures I first explain, in a rather basic fashion, the construction of effective field theories. I then discuss some recent developments in the application of such theories to two- and three-nucleon systems.

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1 Introduction: what is an effective theory?

An effective theory is a *systematic approximation* to some underlying dynamics (which may be known or unknown) that is valid in some **specified regime**. An effective theory is not a model, since its systematic character means that, in principle, predictions of arbitrary accuracy may be made. However, if this is to be true then a small parameter, such as the α of quantum electrodynamics, must govern the systematic approximation scheme. As we shall see here, in many modern effective theories the expansion parameter is a ratio of two physical scales. For instance, in effective theories of supersymmetric physics "beyond the standard model" the ratio would be of p or m, the momentum or mass of a standard model particle, to $M_{\rm SUSY}$. In an effective theory for finite-proton-size effects in the hydrogen atom the small parameter would be $r_{\rm p}$, the proton size, divided by $r_{\rm b}$, the Bohr radius. The smallness of this parameter is then indicative of the domain of validity of the effective theory (ET). In this sense effective theories, like revolutions, carry the seeds of their own destruction, since the failure of the expansion to converge is a signal to the user that he or she is pushing the theory beyond its limits. Within the radius of convergence of the ET the ET "works" because of the following fundamental tenet:

> Phenomena at low energies (or long wavelength) cannot probe details of the high-energy (or short-distance) structure of particles.

I suspect that most physicists subscribe to this tenet: indeed, if the tenet were not true, then physics (other than calculations using a "theory of everything") would be impossible.

In this first section I will begin by discussing the basic ideas of effective theories using a few simple examples from undergraduate physics. In this way we will move from classical effective theories, to classical effective field theories (EFTs), to quantum effective field theories. Of course, to understand the latter one must be able to compute quantum-field-theoretic loop graphs, so this requires a little more education than the standard undergraduate curriculum contains—at least the undergraduate curriculum at American Universities! Nevertheless, I will attempt to present the work at a level that is understandable by someone who has completed a first course on quantum field theory, but has not necessarily yet studied the theory of regularization and renormalization.

In Section 2 I will begin to focus on the nucleon-nucleon (NN) system. This will first necessitate some definitions of terms, notation, and so forth. I then move on to discuss the special issues stemming from the presence of shallow bound states in the NN problem. After displaying one solution to this difficulty, I will define and employ an effective field theory which takes into account the presence of shallow bound states, but other than this only contains neutrons and protons as dynamical degrees of freedom. I will give examples of the success of this EFT, known as EFT($\not\pi$), in computing (very-)low-energy electromagnetic observables in the NN system.

The extension of this work to the three-body problem raises some intriguing problems of renormalization. I will attempt to elucidate these in Section 3, where I draw on the work of Bedaque, Hammer, and van Kolck, to show how, after some thought and interesting discoveries, EFT($\not\pi$) can be applied to the NNN problem.

Finally, in Section 4 I give a brief tour of results in an effective field theory with pions. The effective field theory of QCD in which nucleons and pions are the degrees of freedom is chiral perturbation theory (χPT). We were fortunate to have one of the founders of χPT, and indeed a pioneer in the field of EFTs, lecturing at this school. Prof. Leutwyler's lectures in this volume should be read in conjunction with the material presented here. Indeed, this article should be regarded as little more than light reading on the subject of nuclear EFTs. It is very far from being a thorough review on the topic. The reader who wishes to study the subject in detail should consult the reviews Ref. [1, 2] which contain much more information than does this manuscript. These two reviews also contain full references to the original literature, a job I have not tackled in any systematic way here.

1.1 Some very simple effective theories

1.1.1 Gravity for $h < R$

One of the simplest effective theories I know is one that is learned by high-school physics students. It concerns the standard formula for the gravitational potential-energy difference of an object of mass m which is raised a height h above the Earth's surface:

$$\Delta U = mgh, \tag{1}$$

where g is the acceleration due to gravity

Of course from Newton's Law of Universal Gravitation (itself an effective theory, valid in the limit of small space-time curvature), we have

$$\Delta U = -\frac{GMm}{r_{\rm f}} + \frac{GMm}{r_{\rm i}}, \tag{2}$$

for an object whose distance from the Earth's centre is initially r_i and finally r_f. Now, if we write

$$r_\mathrm{f} = r_\mathrm{i} + h, \tag{3}$$

and assume that $r_\mathrm{i} \approx R$, the radius of the Earth, then

$$\Delta U = m \frac{GM}{R^2} \frac{R}{R+h} h. \tag{4}$$

Identifying $\frac{GM}{R^2} = g$ we see that Eq. (1) is only the first term in a series, which converges as long as $h/R < 1$:

$$\Delta U = mgh \left(1 - \tfrac{h}{R} + \tfrac{h^2}{R^2} + \ldots\right). \tag{5}$$

If we try to apply this theory to a satellite in geosynchronous orbit ($h \gg R$) the series will not converge. But for the space shuttle ($h \sim$ a few hundred km) this series should converge fairly rapidly. Equation (1) is the first term in the effective theory expansion (5) for the gravitational potential energy, with that ET being valid for $h < R$.

1.1.2 Effective theories in the hydrogen atom

Presumably, the hydrogen atom is ultimately described in terms of string theory, or some other fundamental theory of physics. Nevertheless, to very good precision, we can use quantum electrodynamics (QED) as an effective theory to compute its spectrum. The reason why we can ignore corrections to QED from physics at the Planck scale when calculating the hydrogen-atom spectrum will become clear below.

In the case of the hydrogen atom there is an effective theory for QED that is valid up to corrections suppressed by one power of $\alpha = e^2/(4\pi)$, the fine-structure constant. That effective theory is known as the Schrödinger equation with the Coulomb potential. The radial wave function $u_{nl}(r)$ obeys the differential equation [1]): 5

$$-\frac{1}{2m_\mathrm{e}} \frac{d^2 u_{nl}}{dr^2} + \frac{l(l+1)}{r^2} u_{nl}(r) - \frac{\alpha}{r} u_{nl}(r) = E_n u_{nl}(r). \tag{6}$$

The solution, for the lowest-energy eigenstate ($n = 1$; $l = 0$) is, of course:

$$u_{10}(r) = \mathcal{N} \exp(-\alpha m_\mathrm{e} r) = \mathcal{N} \exp(-r/r_\mathrm{b}), \tag{7}$$

where $r_\mathrm{b} = (\alpha m_\mathrm{e})^{-1} \approx (4\ \mathrm{keV})^{-1} \approx 0.5$ Å, and $\mathcal{N}$ is determined by the normalization condition. The corresponding eigenvalue is

$$E_{10} = -\frac{1}{2m_\mathrm{e}} \left(\frac{1}{r_\mathrm{b}}\right)^2. \tag{8}$$

[1]) Throughout I work in units where $\hbar = c = 1$.

The Bohr radius, $r_{\rm b}$, sets the scale for most phenomena associated with the electron in the Hydrogen atom. In particular, the typical momentum of the electron is $\sim 1/r_{\rm b}$, which means that relativistic corrections to the Schrödinger equation are suppressed by $(m_{\rm e} r_{\rm b})^{-2} = \alpha^2$, thereby validating the non-relativistic treatment. Note that if we were discussing muonic hydrogen the energy and distance scales would be very different, since $r_{\rm b}^{\mu} \approx \frac{1}{200} r_{\rm b}^{e}$.

The Bohr radius is large compared to the size of the proton, $r_{\rm p}$, and also compared to the scale of internal structure of the electron. One sense in which the electron has internal structure is that it is dressed by virtual photons. In fact, the Lamb shift is, in fact, just such an electron-structure effect, so "finite-electron-size" effects must be considered if very accurate results are desired. If we for the moment ignore the electron's internal structure and consider only the internal structure of the proton we would replace the Coulomb potential $-\alpha/r$, by the potential generated by an extended proton:

$$V(\mathbf{r}) = -\frac{e^2}{4\pi} \int \frac{\rho(\mathbf{r}')\mathrm{d}^3 r'}{|\mathbf{r} - \mathbf{r}'|}, \tag{9}$$

with $\rho(\mathbf{r}')$ the local electric charge density of the proton at the point $\mathbf{r}'$. Now we make a multipole expansion, in order to obtain:

$$V(\mathbf{r}) = -\frac{e^2}{4\pi r} \sum_{n=0}^{\infty} \left(\frac{r_{\rm p}}{r}\right)^n \int \mathrm{d}^3 r' \, \rho(\mathbf{r}') P_n(\hat{r} \cdot \hat{r}') \left(\frac{r'}{r_{\rm p}}\right)^n \quad \text{for } r > r_{\rm p}, \tag{10}$$

with P_n the nth Legendre polynomial. Here, $\rho(\mathbf{r}')$ only has support for $r' < r_{\rm p}$, and so the integrals are all numbers of order one. Since the solution of the differential equation (6) is mainly sensitive to $r \sim r_{\rm b}$, the expansion parameter here is $r_{\rm p}/r_{\rm b} \sim 1\ \mathring{A}/1$ fm, and so this series converges rapidly, with it entirely permissible to evaluate the corrections for the finite size of the proton in perturbation theory. Nevertheless, an accurate computation requires inclusion of the term of order $(r_{\rm p}/r_{\rm b})^2$ in this expansion.

In fact, this term of $O((r_{\rm p}/r_{\rm b})^2)$ is the first correction due to finite-size effects in Eq. (6). This is easily seen from Eq. (10), since the coefficient of the term of $O(r_{\rm p}/r_{\rm b})$ is zero, as long as the proton's charge distribution is even under parity. Thus consideration of the scales in the problem alone would lead us to grossly overestimate the magnitude of the finite-size effect. It is the combination of scales and symmetry that leads to an accurate estimate of the magnitude of the effects neglected by assuming that the proton is point-like in Eq. (6). These two principles:

– a ratio of scales generating an expansion parameter,

– symmetries constraining the types of corrections that can appear,

inform the construction of most effective theories.

1.2 Building a (classical) effective theory: the scattering of light from atoms

Next I want to discuss an example of ET-construction which first appeared in print in the effective field theory lecture notes of Kaplan [3]. These lecture notes are an excellent introduction to EFT, and this example provides a great demonstration of ET construction. Here I have reworked some of the notation, but the basic idea is as in Ref. [3].

Consider the scattering of low-energy light from an atom. The question we must answer is: What is the Hamiltonian that describes the interaction of the atom with the electromagnetic field of the incoming light? To do this, we first consider the scales in the problem: the energy of the electromagnetic field, ω is assumed small compared to the spacing of the atomic levels, ΔE, and the inverse size of the atom. We will assume in turn that all of these scales are much smaller than the mass of the atom. Using $r_{\rm b}$ to estimate ΔE we have the following hierarchy of scales:

$$\omega \ll \Delta E \sim \frac{1}{m_{\rm e} r_{\rm b}^2} \ll \frac{1}{r_{\rm b}} \ll M_{\rm atom}. \tag{11}$$

Meanwhile the symmetries of the theory will be electromagnetic gauge invariance $[U(1)_{\rm em}]$, rotational invariance, and Hermitian conjugation/Time reversal. These symmetries constrain the types of terms that we can write in our Hamiltonian. Firstly, the constraint of gauge invariance suggests that it is wise to construct $H_{\rm atom}$ out of the quantities $\mathbf{E}$ and $\mathbf{B}$, rather than using the four-vector potential A_μ. Then, secondly, rotational invariance suggests that we employ quantities such as $\mathbf{E}^2$ and $\mathbf{E}\cdot\mathbf{B}$ in $H_{\rm atom}$. However, $\mathbf{E}\cdot\mathbf{B}$ is odd under time reversal, and so we cannot write down a term proportional to it in $H_{\rm atom}$. Meanwhile, terms such as $\nabla\cdot\mathbf{B}$, and $\nabla\times\mathbf{E}$ may be included in $H_{\rm atom}$, but then can be eliminated from the Hamiltonian using the field equations for the electromagnetic field in the region around the atom:

$$\nabla\cdot\mathbf{E} = 0; \qquad \nabla\cdot\mathbf{B} = 0; \tag{12}$$

$$\nabla\times\mathbf{E} = 0; \qquad \nabla\times\mathbf{B} = 0. \tag{13}$$

This leaves us with:

$$H_{\rm atom} = a_1\mathbf{B}^2 + a_2\mathbf{E}^2 + a_3(\partial_0\mathbf{B}^2) + a_4(\partial_0\mathbf{E}^2) + a_5(\mathbf{E}\cdot\mathbf{B})^2 + \ldots \tag{14}$$

In spite of our cleverness in constraining the terms that may appear, we are still left with infinitely many operators that can contribute to $H_{\rm atom}$. How are we to organize all of these contributions?

Interlude: naive dimensional analysis

The answer lies in the scale hierarchy established in Eq. (11), together with a simple technique known as naive dimensional analysis (NDA). This works as follows: consider, for instance, the operator $\mathbf{B}^2$. Counting powers of energy/momentum we see that it carries four powers of energy, which we write as:

$$[\mathbf{B}^2] = 4. \tag{15}$$

However, $H_{\rm atom}$ must have dimensions of energy. It follows that a_1 and a_2 must each carry three negative powers of energy/momentum:

$$[a_1] = [a_2] = -3, \tag{16}$$

that is to say:

$$a_1, a_2 = \frac{1}{(\text{some energy scale})^3}. \tag{17}$$

Now we ask what energy scales are present in the problem and so might appear in the denominator here. The photon energy ω cannot appear in the denominator since the scale that occurs there should refer to a property of the atom. Any of the scales r_0^{-1}, ΔE, or $M_{\rm atom}$ could be involved though. The most conservative estimate would be that:

$$a_1, a_2 \sim \frac{1}{(\Delta E)^3}. \tag{18}$$

However, very low-energy photons cannot probe the quantum level structure of the atom: they should interact with the entire atom in an essentially classical way. Thus, ΔE cannot occur in the denominator of this lowest-dimensional term in the Hamiltonian, and so we deduce that a_1 and a_2 must scale with r_0, i. e.:

$$a_1, a_2 \sim r_0^3, \tag{19}$$

where the $\sim$ usually indicates that the coefficient here could be a 3 or a 1/3 (or a -3 or a -1/3) but will generally be a number of order one [2]). a_1 and a_2 are in fact proportional to the electric and magnetic polarizabilities of the atom [4].

Meanwhile, the operators multiplying the coefficients a_3 and a_4 have dimension 5. Thus, $[a_3] = [a_4] = -4$, and so a_3 and a_4 carry one more energy-scale downstairs as compared to a_1 and a_2. Conservatively, we assign the scaling:

$$a_3, a_4 \sim \frac{r_0^3}{\Delta E}. \tag{20}$$

Similar estimates can be made for the other terms in $H_{\rm atom}$. The key point is that since $\mathbf{B}^2$ and $\mathbf{E}^2$ are the lowest dimension operators allowed by the symmetries and not already constrained by field equations, they will give the dominant effect in $H_{\rm atom}$ for low-energy photons. Any higher-order effects will be suppressed by at least $\omega/\Delta E$, i. e.:

$$H_{\rm atom} = r_0^3 \left[\tilde{a_1}\mathbf{B}^2 + \tilde{a_2}\mathbf{E}^2 + O\left(\frac{\omega}{\Delta E}\right)\right], \tag{21}$$

where $\tilde{a_1}$ and $\tilde{a_2}$ are now dimensionless numbers. It is straightforward to turn this result into a prediction for the photon-atom cross section. Since $[\sigma] = -2$ and the

[2]) There is a subtlety here: since this is an electromagnetic interaction the argument here suffices to get the scaling with ω correct, but it does not count powers of $\alpha_{\rm em} = 1/137$ which is an additional small parameter in the problem.

cross section results from squaring the quantum-mechanical amplitude arising from the Hamiltonian (21) we discover that

$$\sigma \sim \omega^4 r_0^6 \left[1 + O\left(\frac{\omega}{\Delta E}\right)\right], \tag{22}$$

where the $\sim$ disguises the hard work needed to figure out all the factors of 2, π and so forth that really go into deriving σ! The strong dependence of σ on ω is, of course, the reason the sky is blue—as pointed out in Ref. [3] or in Ref. [5], where the constant of proportionality in Eq. (22) is worked out in detail!

1.3 A classical effective *field* theory: Fermi electroweak theory

Equation (21) is an effective expression for the classical Hamiltonian that is valid at long wavelength, or equivalently, for low-energy electromagnetic fields. In general effective field theories are derived for low energies, although this need not be the case.

A canonical example of a low-energy effective field theory is Fermi's electroweak theory. This is an effective field theory that can be used to compute, say, low-energy electron-neutrino scattering. The only particles explicitly appearing in this theory are electrons and neutrinos. By contrast, in the standard model, the electrons and neutrinos interact by the exchange of W and Z bosons. If we wish to compute the scattering of neutrinos from electrons we could compute the full standard model amplitude for diagrams such as Fig. 1 [6]:

$$\mathcal{A} = \left(\frac{-\mathrm{i}g}{2\cos\theta_\mathrm{W}}\right)(\bar{\nu}P_\mathrm{L}\gamma_\mu\nu)\,\frac{\mathrm{i}}{q^2 - M_\mathrm{Z}^2}\left(\frac{-\mathrm{i}g}{2\cos\theta_\mathrm{W}}\right)(\bar{e}\gamma^\mu Q e)\,; \tag{23}$$

with:

$$P_\mathrm{L,R} = \tfrac{1}{2}(1 \mp \gamma_5); \tag{24}$$

$$Q = (-1 + 2\sin^2\theta_\mathrm{W})P_\mathrm{L} + 2\sin^2\theta_\mathrm{W} P_\mathrm{R}. \tag{25}$$

and $q = p' - p$ being the change in momentum of the neutrino. Relating q^2 to laboratory quantities, we see that:

$$q^2 = -4E_\mathrm{lab}E'_\mathrm{lab}\sin^2\left(\frac{\theta_\mathrm{lab}}{2}\right), \tag{26}$$

where E_lab (E'_lab) and θ_lab are the initial (final) energy and scattering angle of the neutrino in the lab. system. So, if $E_\mathrm{lab} \ll M_\mathrm{Z}$, then we can expand the Z-propagator in Taylor series. The leading term in this series is then:

$$\mathcal{A} = \mathrm{i}\sqrt{2}G_\mathrm{F}\,(\bar{\nu}P_\mathrm{L}\gamma_\mu\nu)\,(\bar{e}\gamma^\mu Q e)\,, \tag{27}$$

with:

$$G_\mathrm{F} = \frac{g^2}{4\sqrt{2}\cos^2\theta_\mathrm{W}}\frac{1}{M_\mathrm{Z}^2} = \frac{g^2}{4\sqrt{2}M_\mathrm{W}^2}. \tag{28}$$