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## Chapter IV.1

# Symmetry Breaking

### A symmetric world would be dull

While we would like to believe that the fundamental laws of Nature are symmetric, a completely symmetric world would be rather dull, and as a matter of fact, the real world is not perfectly symmetric. More precisely, we want the Lagrangian, but not the world described by the Lagrangian, to be symmetric. Indeed, a central theme of modern physics is the study of how symmetries of the Lagrangian can be broken. We will see in subsequent chapters that our present understanding of the fundamental laws is built upon an understanding of symmetry breaking.

Consider the Lagrangian studied in Chapter I.9:

$$\mathcal{L} = \frac{1}{2} \left[ (\partial\vec{\varphi})^2 - \mu^2\vec{\varphi}^2 \right] - \frac{\lambda}{4} (\vec{\varphi}^2)^2 \tag{1}$$

where  $\vec{\varphi} = (\varphi_1, \varphi_2, \dots, \varphi_N)$ . This Lagrangian exhibits an  $O(N)$  symmetry under which  $\vec{\varphi}$  transforms as an  $N$ -component vector.

We can easily add terms that do not respect the symmetry. For instance, add terms such as  $\varphi_1^2$ ,  $\varphi_1^4$  and  $\varphi_1^2\vec{\varphi}^2$  and break the  $O(N)$  symmetry down to  $O(N - 1)$ , under which  $\varphi_2, \dots, \varphi_N$  rotate as an  $(N - 1)$ -component vector. This way of breaking the symmetry, “by hand” as it were, is known as explicit breaking.

We can break the symmetry in stages. Obviously, if we want to, we can break it down to  $O(N - M)$  by hand, for any  $M < N$ .

Note that in this example, with the terms we added, the reflection symmetry  $\varphi_a \rightarrow -\varphi_a$  (any  $a$ ) still holds. It is easy enough to break this symmetry as well, by adding a term such as  $\varphi_a^3$ , for example.

Breaking the symmetry by hand is not very interesting. Indeed, we might as well start with a nonsymmetric Lagrangian in the first place.

### Spontaneous symmetry breaking

A more subtle and interesting way is to let the system “break the symmetry itself,” a phenomenon known as spontaneous symmetry breaking. I will explain by way of an example. Let us flip the sign of the  $\vec{\varphi}^2$  term in (1) and write

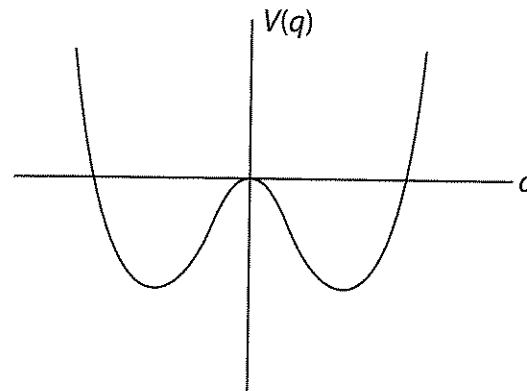


Figure IV.1.1

$$\mathcal{L} = \frac{1}{2} \left[ (\partial \bar{\varphi})^2 + \mu^2 \bar{\varphi}^2 \right] - \frac{\lambda}{4} (\bar{\varphi}^2)^2 \quad (2)$$

Naively, we would conclude that for small  $\lambda$  the field  $\varphi$  creates a particle of mass  $\sqrt{-\mu^2} = i\mu$ . Something is obviously wrong.

The essential physics is analogous to what would happen if we give the spring constant in an anharmonic oscillator the wrong sign and write  $L = \frac{1}{2}(\dot{q}^2 + kq^2) - (\lambda/4)q^4$ . We all know what to do in classical mechanics. The potential energy  $V(q) = -\frac{1}{2}kq^2 + (\lambda/4)q^4$  [known as the double-well potential (Figure IV.1.1)] has two minima at  $q = \pm v$ , where  $v \equiv (k/\lambda)^{1/2}$ . At low energies, we choose either one of the two minima and study small oscillations around that minimum. Committing to one or the other of the two minima breaks the reflection symmetry  $q \rightarrow -q$  of the system.

In quantum mechanics, however, the particle can tunnel between the two minima, the tunneling barrier being  $V(0) - V(\pm v)$ . The probability of being in one or the other of the two minima must be equal, thus respecting the reflection symmetry  $q \rightarrow -q$  of the Hamiltonian. In particular, the ground state wave function  $\psi(q) = \psi(-q)$  is even.

Let us try to extend the same reasoning to quantum field theory. For a generic scalar field Lagrangian  $\mathcal{L} = \frac{1}{2}(\partial_0\varphi)^2 - \frac{1}{2}(\partial_i\varphi)^2 - V(\varphi)$  we again have to find the minimum of the potential energy  $\int d^Dx [\frac{1}{2}(\partial_i\varphi)^2 + V(\varphi)]$ , where  $D$  is the dimension of space. Clearly, any spatial variation in  $\varphi$  only increases the energy, and so we set  $\varphi(x)$  to equal a spacetime independent quantity  $\varphi$  and look for the minimum of  $V(\varphi)$ . In particular, for the example in (2), we have

$$V(\varphi) = -\frac{1}{2}\mu^2\bar{\varphi}^2 + \frac{\lambda}{4}(\bar{\varphi}^2)^2 \quad (3)$$

As we will see, the  $N = 1$  case is dramatically different from the  $N \geq 2$  cases.

### Difference between quantum mechanics and quantum field theory

Study the  $N = 1$  case first. The potential  $V(\varphi)$  looks exactly the same as the potential in Figure IV.1.1 with the horizontal axis relabeled as  $\varphi$ . There are two minima at  $\varphi = \pm v = \pm(\mu^2/\lambda)^{1/2}$ .

But some thought reveals a crucial difference between quantum field theory and quantum mechanics. The tunneling barrier is now  $[V(0) - V(\pm v)] \int d^Dx$  (where  $D$  denotes the dimension of space) and hence infinite (or more precisely, extensive with the volume of the system)! Tunneling is shut down, and the ground state wave function is concentrated around either  $+v$  or  $-v$ . We have to commit to one or the other of the two possibilities for the ground state and build perturbation theory around it. It does not matter which one we choose: The physics is equivalent. But by making a choice, we break the reflection symmetry  $\varphi \rightarrow -\varphi$  of the Lagrangian.

The reflection symmetry is broken spontaneously! We did not put symmetry breaking terms into the Lagrangian by hand but yet the reflection symmetry is broken.<sup>1</sup>

Let's choose the ground state at  $+v$  and write  $\varphi = v + \varphi'$ . Expanding in  $\varphi'$  we find after a bit of arithmetic that

$$\mathcal{L} = \frac{\mu^4}{4\lambda} + \frac{1}{2}(\partial\varphi')^2 - \mu^2\varphi'^2 - O(\varphi'^3) \quad (4)$$

The physical particle created by the shifted field  $\varphi'$  has mass  $\sqrt{2}\mu$ . The physical mass squared has to come out positive since, after all, it is just  $V''(\varphi)|_{\varphi=v}$ , as you can see after a moment's thought.

Similarly, you would recognize that the first term in (4) is just  $V(\varphi)|_{\varphi=v}$ . If we are only interested in the scattering of the mesons associated with  $\varphi'$  this term does not enter at all. Indeed, we are always free to add an arbitrary constant to  $\mathcal{L}$  to begin with. We had quite arbitrarily set  $V(\varphi = 0)$  equal to 0. The same situation appears in quantum mechanics: In the discussion of the harmonic oscillator the zero point energy  $\frac{1}{2}\hbar\omega$  is not observable; only transitions between energy levels are physical. We will return to this point in Chapter VIII.2.

Yet another way of looking at (2) is that quantum field theory amounts to doing the Euclidean functional integral

$$Z = \int D\varphi e^{-\int d^Dx [\frac{1}{2}(\partial\varphi)^2 - \mu^2\varphi^2 + \frac{\lambda}{4}(\varphi^2)^2]}$$

and perturbation theory just corresponds to studying the small oscillations around a minimum of the Euclidean action. Normally, with the sign of  $\varphi^2$  in  $\mathcal{L}$  negative,

<sup>1</sup> An insignificant technical aside for the nitpickers: Strictly speaking, in field theory the ground state wave function should be called a wave functional, since  $\Psi[\varphi(\vec{x})]$  is a functional of the function  $\varphi(\vec{x})$ .

we expand around the minimum  $\varphi = 0$ . With the sign of  $\varphi^2$  negative,  $\varphi = 0$  is a local maximum and not a minimum.

In quantum field theory what is called the ground state is also known as the vacuum, since it is literally the state in which the field is “at rest,” with no particles present. Here we have two physically equivalent vacua from which we are to choose one. The value assumed by  $\varphi$  in the ground state, either  $v$  or  $-v$  in our example, is known as the vacuum expectation value of  $\varphi$ . The field  $\varphi$  is said to have acquired a vacuum expectation value.

### Continuous symmetry

Let us now turn to (2) with  $N \geq 2$ . The potential (3) is shown in Figure IV.1.2 for  $N = 2$ . The shape of the potential has been variously compared to the bottom of a punted wine bottle or a Mexican hat. The potential is minimized at  $\vec{\varphi}^2 = \mu^2/\lambda$ . Something interesting is going on: We have an infinite number of vacua characterized by the direction of  $\vec{\varphi}$  in that vacuum. Because of the  $O(2)$  symmetry of the Lagrangian they are all physically equivalent. The result had better not depend on our choice. So let us choose  $\vec{\varphi}$  to point in the 1 direction, that is,  $\varphi_1 = v \equiv +\sqrt{\mu^2/\lambda}$  and  $\varphi_2 = 0$ .

Now consider fluctuations around this field configuration, in other words, write  $\varphi_1 = v + \varphi'_1$  and  $\varphi_2 = \varphi'_2$ , plug into (2) for  $N = 2$ , and expand  $\mathcal{L}$  out. I invite you to do the arithmetic. You should find (after dropping the primes on the fields; why clutter the notation, right?)

$$\mathcal{L} = \frac{\mu^4}{4\lambda} + \frac{1}{2} \left[ (\partial\varphi_1)^2 + (\partial\varphi_2)^2 \right] - \mu^2\varphi_1^2 + O(\varphi^3) \quad (5)$$

The constant term is exactly as in (4), and just like the field  $\varphi'$  in (4), the field  $\varphi_1$  has mass  $\sqrt{2}\mu$ . But now note the remarkable feature of (5): the absence of a  $\varphi_2^2$  term. The field  $\varphi_2$  is massless!

### Emergence of massless boson

That  $\varphi_2$  comes out massless is not an accident. I will now explain that the masslessness is a general and exact phenomenon.

Referring back to Figure IV.1.2 we can easily understand the particle spectrum. Excitation in the  $\varphi_1$  field corresponds to fluctuation in the radial direction, “climbing the wall” so to speak, while excitation in the  $\varphi_2$  field corresponds to fluctuation in the angular direction, “rolling along the gutter” so to speak. It costs no energy for a marble to roll along the minima of the potential energy, going from one minimum to another. Another way of saying this is to picture a long wavelength excitation of the form  $\varphi_2 = a \sin(\omega t - \vec{k}\vec{x})$  with  $a$  small. In a region of length scale small compared to  $|\vec{k}|^{-1}$ , the field  $\varphi_2$  is essentially constant and thus the field  $\vec{\varphi}$  is just rotated slightly away from the 1 direction, which by the  $O(2)$  symmetry is equivalent to

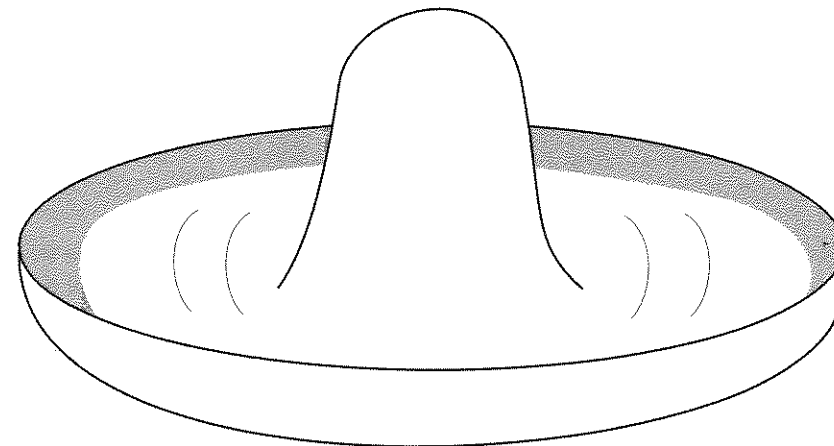


Figure IV.1.2

the vacuum. It is only when we look at regions of length scale large compared to  $|\vec{k}|^{-1}$  that we realize that the excitation costs energy. Thus, as  $|\vec{k}| \rightarrow 0$ , we expect the energy of the excitation to vanish.

We now understand the crucial difference between the  $N = 1$  and the  $N = 2$  cases: In the former we have a reflection symmetry, which is discrete, while in the latter we have an  $O(2)$  symmetry, which is continuous.

We have worked out the  $N = 2$  case in detail. You should now be able to generalize our discussion to arbitrary  $N \geq 2$  (see Exercise (IV.1.1)).

Meanwhile, it is worth looking at  $N = 2$  from another point of view. Many field theories can be written in more than one form and it is important to know them under different guises. Construct the complex field  $\varphi = (1/\sqrt{2})(\varphi_1 + i\varphi_2)$ ; we have  $\varphi^\dagger\varphi = \frac{1}{2}(\varphi_1^2 + \varphi_2^2)$  and so can write (2) as

$$\mathcal{L} = \partial\varphi^\dagger\partial\varphi + \mu^2\varphi^\dagger\varphi - \lambda(\varphi^\dagger\varphi)^2 \quad (6)$$

which is manifestly invariant under the  $U(1)$  transformation  $\varphi \rightarrow e^{i\alpha}\varphi$  (recall Chapter I.9). You may recognize that this amounts to saying that the groups  $O(2)$  and  $U(1)$  are locally isomorphic. Just as we can write a vector in Cartesian or polar coordinates we are free to parametrize the field by  $\varphi(x) = \rho(x)e^{i\theta(x)}$  (as in Chapter III.5) so that  $\partial_\mu\varphi = (\partial_\mu\rho + i\rho\partial_\mu\theta)e^{i\theta}$ . We obtain  $\mathcal{L} = \rho^2(\partial\theta)^2 + (\partial\rho)^2 + \mu^2\rho^2 - \lambda\rho^4$ . Spontaneous symmetry breaking means setting  $\rho = v + \chi$  with  $v = +\sqrt{\mu^2/2\lambda}$ , whereupon

$$\mathcal{L} = v^2(\partial\theta)^2 + \left[ (\partial\chi)^2 - 2\mu^2\chi^2 - 4\sqrt{\frac{\mu^2\lambda}{2}}\chi^3 - \lambda\chi^4 \right] + \left( \sqrt{\frac{2\mu^2}{\lambda}}\chi + \chi^2 \right) (\partial\theta)^2 \quad (7)$$

We recognize the phase  $\theta(x)$  as the massless field. We have arranged the terms in the Lagrangian in three groups: the kinetic energy of the massless field  $\theta$ , the kinetic and potential energy of the massive field  $\chi$ , and the interaction between  $\theta$  and  $\chi$ .

### Goldstone's theorem

We will now prove Goldstone's theorem, which states that whenever a continuous symmetry is spontaneously broken, massless fields, known as Nambu-Goldstone bosons, emerge.

Recall that associated with every continuous symmetry is a conserved charge  $Q$ . That  $Q$  generates a symmetry is stated as

$$[H, Q] = 0 \quad (8)$$

Let the vacuum (or ground state in quantum mechanics) be denoted by  $|0\rangle$ . By adding an appropriate constant to the Hamiltonian  $H \rightarrow H + c$  we can always write  $H|0\rangle = 0$ . Normally, the vacuum is invariant under the symmetry transformation,  $e^{i\theta Q}|0\rangle = |0\rangle$ , or in other words  $Q|0\rangle = 0$ .

But suppose the symmetry is spontaneously broken, so that the vacuum is not invariant under the symmetry transformation; in other words,  $Q|0\rangle \neq 0$ . Consider the state  $Q|0\rangle$ . What is its energy? Well,

$$HQ|0\rangle = [H, Q]|0\rangle = 0 \quad (9)$$

[The first equality follows from  $H|0\rangle = 0$  and the second from (8).] Thus, we have found another state  $Q|0\rangle$  with the same energy as  $|0\rangle$ .

Note that the proof makes no reference to either relativity or fields. You can also see that it merely formalizes the picture of the marble rolling along the gutter.

In quantum field theory, we have local currents, and so

$$Q = \int d^D x J^0(\vec{x}, t)$$

where  $D$  denotes the dimension of space and conservation of  $Q$  says that the integral can be evaluated at any time. Consider the state

$$|s\rangle = \int d^D x e^{i\vec{k}\vec{x}} J^0(\vec{x}, t) |0\rangle$$

which has<sup>2</sup> spatial momentum  $\vec{k}$ . As  $\vec{k}$  goes to zero it goes over to  $Q|0\rangle$ , which as we learned in (9) has zero energy. Thus, as the momentum of the state  $|s\rangle$  goes to zero, its energy goes to zero. In a relativistic theory, this means precisely that  $|s\rangle$  describes a massless particle.

The proof makes clear that the theorem practically exudes generality: It applies to any spontaneously broken continuous symmetry.

### Counting Nambu-Goldstone bosons

From our proof, we see that the number of Nambu-Goldstone bosons is clearly equal to the number of conserved charges that do not leave the vacuum invariant, that is, do not annihilate  $|0\rangle$ . For each such charge  $Q^\alpha$ , we can construct a zero-energy state  $Q^\alpha|0\rangle$ .

In our example, we have only one current  $J_\mu = i(\varphi_1\partial_\mu\varphi_2 - \varphi_2\partial_\mu\varphi_1)$  and hence one Nambu-Goldstone boson. In general, if the Lagrangian is left invariant by a symmetry group  $G$  with  $n(G)$  generators, but the vacuum is left invariant by only a subgroup  $H$  of  $G$  with  $n(H)$  generators, then there are  $n(G) - n(H)$  Nambu-Goldstone bosons. If you want to show off your mastery of mathematical jargon you can say that the Nambu-Goldstone bosons live on the coset space  $G/H$ .

### Ferromagnet and spin wave

The generality of the proof suggests that the usefulness of Goldstone's theorem is not restricted to particle physics. In fact, it originated in condensed matter physics, the classic example there being the ferromagnet. The Hamiltonian, being composed of just the interaction of nonrelativistic electrons with the ions in the solid, is of course invariant under the rotation group  $SO(3)$ , but the magnetization  $\vec{M}$  picks out a direction, and the ferromagnet is left invariant only under the subgroup  $SO(2)$  consisting of rotations about the axis defined by  $\vec{M}$ . The Nambu-Goldstone theorem is easy to visualize physically. Consider a "spin wave" in which the local magnetization  $\vec{M}(\vec{x})$  varies slowly from point to point. A physicist living in a region small compared to the wavelength does not even realize that he or she is no longer in the "vacuum." Thus, the frequency of the wave must go to zero as the wavelength goes to infinity. This is of course exactly the same heuristic argument given earlier. Note that quantum mechanics is needed only to translate the wave

<sup>2</sup> Acting on it with  $P^i$  (Exercise I.10.3) and using  $P^i|0\rangle = 0$ , we have

$$P^i|s\rangle = \int d^D x e^{i\vec{k}\vec{x}} [P^i, J^0(\vec{x}, t)]|0\rangle = k^i|s\rangle$$

upon integrating by parts.

vector  $\vec{k}$  into momentum and the frequency  $\omega$  into energy. I will come back to magnets and spin wave in Chapters V.3 and VI.5.

### Quantum fluctuations and the dimension of spacetime

Our discussion of spontaneous symmetry breaking is essentially classical. What happens when quantum fluctuations are included? I will address this question in detail in Chapter IV.3, but for now let us go back to (5). In the ground state,  $\varphi_1 = v$  and  $\varphi_2 = 0$ . Recall that in the mattress model of a scalar field theory the mass term comes from the springs holding the mattress to its equilibrium position. The term  $-\mu^2 \varphi_1'^2$  (note the prime) in (5) tells us that it costs action for  $\varphi_1'$  to wander away from its ground state value  $\varphi_1' = 0$ . But now we are worried:  $\varphi_2$  is massless. Can it wander away from its ground state value? To answer this question let us calculate the mean square fluctuation

$$\begin{aligned} \langle (\varphi_2(0))^2 \rangle &= \frac{1}{Z} \int D\varphi e^{iS(\varphi)} [\varphi_2(0)]^2 \\ &= \lim_{x \rightarrow 0} \frac{1}{Z} \int D\varphi e^{iS(\varphi)} \varphi_2(x) \varphi_2(0) \\ &= \lim_{x \rightarrow 0} \int \frac{d^d k}{(2\pi)^d} \frac{e^{ikx}}{k^2} \end{aligned} \quad (10)$$

(We recognized the functional integral that defines the propagator; recall Chapter I.7.) The upper limit of the integral in (10) is cut off at some  $\Lambda$  (which would correspond to the inverse of the lattice spacing when applying these ideas to a ferromagnet) and so as explained in Chapter III.1 (and as you will see in Chapter VIII.3) we are not particularly worried about the ultraviolet divergence for large  $k$ . But we do have to worry about a possible infrared divergence for small  $k$ . (Note that for a massive field  $1/k^2$  in (10) would have been replaced by  $1/(k^2 + \mu^2)$  and there would be no infrared divergence.)

We see that there is no infrared divergence for  $d > 2$ . Our picture of spontaneously breaking a continuous symmetry is valid in our  $(3 + 1)$ -dimensional world.

However, for  $d \leq 2$  the mean square fluctuation of  $\varphi_2$  comes out infinite, so our naive picture is totally off. We have arrived at the Coleman-Mermin-Wagner theorem (proved independently by a particle theorist and two condensed matter theorists), which states that spontaneous breaking of a continuous symmetry is impossible for  $d = 2$ . Note that while our discussion is given for  $O(2)$  symmetry the conclusion applies to any continuous symmetry since the argument depends only on the presence of Nambu-Goldstone fields.

In our examples, symmetry is spontaneously broken by a scalar field  $\varphi$ , but nothing says that the field  $\varphi$  must be elementary. In many condensed matter systems, superconductors, for example, symmetries are spontaneously broken, but we know that the system consists of electrons and atomic nuclei. The field

$\varphi$  is generated dynamically, for example as a bound state of two electrons in superconductors. More on this in Chapter V.4. The spontaneous breaking of a symmetry by a dynamically generated field is sometimes referred to as dynamical symmetry breaking.<sup>3</sup>

### Exercises

- IV.1.1. Show explicitly that there are  $N - 1$  Nambu-Goldstone bosons in the  $G = O(N)$  example (2).
- IV.1.2. Construct the analog of (2) with  $N$  complex scalar fields and invariant under  $SU(N)$ . Count the number of Nambu-Goldstone bosons when one of the scalar fields acquires a vacuum expectation value.

<sup>3</sup>This chapter is dedicated to the memory of the late Jorge Swieca.