[1] W. R. French and R. L. Bunting, Am. J. Phys. **37**, 637 (1969).
[2] Isotope Products Laboratory, (Burbank, CA) Model FF-252; AERE Harwell, Isotope Sales (Oxfordshire, England). (Note: Coincidence experiments require an ultrathin backing and $\gtrsim 10^3$ fissions/sec, i.e., $\gtrsim 1$ μCi total activity.)
[3] L. A. Turner, Rev. Mod. Phys. **12**, 1 (1940); O. Hahn, Sci. Am. **198**, 76 (1958); E. B. Sparberg, Am. J. Phys. **32**, 2 (1964); H. G. Graetzer, Am. J. Phys. **32**, 9 (1964).
[4] L. Meitner and O. R. Frisch, Nature **143**, 239, 471 (1939).
[5] N. Bohr and J. A. Wheeler, Phys. Rev. **56**, 426 (1939); D. L. Hill and J. A. Wheeler, Phys. Rev. **89**, 1102 (1953); J. A. Wheeler and I. G. Schroder, in *Handbook of Physics*, 2nd ed., edited by E. Condon and H. Odishaw (McGraw-Hill, New York, 1967).
[6] E. K. Hyde, *The Nuclear Properties of the Heavy Elements, Vols. I-III* (Prentice-Hall, Engelwood Cliffs, NJ, 1964).
[7] R. Vandenbosch and J. R Huizenga, *Nuclear Fission* (Academic, New York, 1973) and reference cited therein.
[8] E. Segre, *Nuclei and Particles*, 2nd ed. (Benjamin, Reading, MA, 1977), Chaps. 3,6,7,11, and 12.
[9] K. A. Petrzhak and G. N. Flerov, C. R. Akad. Sci. USSR **25**, 500 (1940).
[10] W. Swiatecki, Phys. Rev. **100**, 936 (1955); **101**, 97 (1956).
[11] W. D. Meyers and W. J. Swiatecki, Nucl. Phys. **81**, 1 (1966).
[12] S. Cohen and W. J. Swiatecki, Ann. Phys. **19**, 67 (1962); **22**, 406 (1963).
[13] W. D. Meyers and W. J. Swiatecki, Ann. Phys. **55**, 395 (1969); **84**, 186 (1974).
[14] J. R. Nix, Ann. Phys. **41**, 52 (1967).
[15] V. M. Strutinsky, Nucl. Phys. A **95**, 420 (1967); **122**, 1 (1968).
[16] S. G. Nilsson *et al.*, Nucl. Phys. A **131**, 1 (1969).
[17] D. N. Schramm and W. A. Fowler, Nature **231**, 103 (1971).
[18] *Table of Isotopes*, 7th ed., edited by C. M. Lederer and V. S. Shirley (Wiley-Interscience, New York, 1978).
[19] W. J. Price, *Nuclear Radiation Detection*, 2nd ed. (McGraw-Hill, New York, 1964).
[20] G. F. Knoll, *Radiation Detection and Measurment* (Wiley, New York, 1979).
[21] *Condensed Product and Application Information* (ORTEC, Inc., Oak Ridge, TN, 1980; Experiments in Nuclear Science-AN 34; Application Note AN40:Heavy Ion Detection (ORTEC, Inc. Oak Ridge, TN; Laboratory Manual for Nuclear Sciences (Canberra, Inc., Meriden, CT, 1978) (unpublished).
[22] LeCroy, Inc., (Spring Valley, NY) Type 100 series IC amplifiers.
[23] J. Ying, Honors thesis, University of Michigan, 1980 (unpublished).
[24] H. W. Schmitt *et al.*, Nucl. Inst. Meth. **40**, 204 (1966); Phys. Rev. **141**, 1146 (1966); V. Forgue and S. Kahn, Nucl. Inst. Meth. **48**, 93 (1967).
[25] S. B. Kaufman *et al.*, Nucl. Inst. Meth. **115**, 47 (1974); P. David *et al.*, Phys. Lett. **60B**, 445 (1976); H. Wohlfarth *et al.*, Nucl. Inst. Meth. **140**, 189 (1977).
[26] W. Booth and I. S. Grant, Nucl. Phys. **63**, 481 (1965) [Note: Eq. (4): x^3 term should be 1131 x^3]; J. B. Marion and F. C. Young, *Nuclear Reaction Analysis* (American Elsvier, New York, 1968), Sec. A II.
[27] Arizona Carbon Foil Co., (Carbon Foils); E. I. Dupont, Inc. (Mylar); P. J. Schweitzer Div. Kimberly-Clark Corp. (KIMFOL polycarbonate); Chromium Corp. of America (nickel and metal foils).
[28] H. Heckman *et al.*, Phys. Rev. **117**, 544 (1960).
[29] L. C. Northcliffe, Ann. Rev. Nucl. Sci. **13**, 67 (1963).
[30] L. C. Northcliffe and R. F. Schilling, Nucl. Data Tables **7**, 233 (1970).
[31] H. D. Betz, Rev. Mod. Phys. **44**, 465 (1972); B. Delaunay, Nucl. Inst. Meth. **146**, 101 (1977).
[32] W. Barkas, *Nuclear Research Emulsions* (Academic, New York, 1973); R. Katz and F. E. Pinkerton, Nucl. Inst. Meth. **130**, 105 (1975).
[33] R. L. Fleischer, P. B. Price, and R. M. Walker, *Nuclear Tracks in Solids* (University of California, Berkeley, CA, 1975).
[34] B. B. Rossi and H. H. Staub, *Ionization Chambers and Counters* (McGraw-Hill, New York, 1949); R. G. Marcley, Am. J. Phys. **30**, 845 (1962).
[35] J. C. Patti and A. Salam, Phys. Rev. Lett. **31**, 661 (1973); F. Reines and M. F. Crouch, Phys. Rev. Lett. **32**, 493 (1974).
[36] H. A. Smith *et al.*, Am. J. Phys. **47**, 71 (1979).

Asymptotic freedom as a spin effect

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It is shown how both the qualitative and the quantitative features of the asymptotic freedom of quantum chromodynamics can be understood in a rather intuitive way. The starting point is the spin of the gluon, which because of the gluon self-coupling makes the vacuum behave like a paramagnetic substance. Combining this result with Lorentz invariance, we conclude that the vacuum exhibits dielectric antiscreening and hence asymptotic freedom. The calculational techniques are with some minor modifications those of the Landau theory on the diamagnetic properties of a free-electron gas.

I. INTRODUCTION

A turning point in recent attempts to formulate a fundamental theory of the strong interactions of elementary particles was reached through the formulation of QCD (quantum chromodynamics). In this theory, strongly interacting elementary particles (protons, pions, kaons, etc.) are constituted of fractionally charged quarks, interacting with each other through exchange of massless vector particles called gluons.

The theory is remarkable because it is asymptotically free,[1] i.e., the effective strength of the interaction decreases at short distances. The reason for this property of the theory is that gluons interact directly with other gluons without the mediation of quarks. To carry through a formal proof of this one has to use a complicated formal device known as "the renormalization group." An intuitive argument without the use of this formalism was recently given in this Journal.[2] In the present paper, an attempt will be done to show that a rather simple and intuitive argument can explain also the quantitative features of asymptotic freedom.

The argument uses the fact that the vacuum of QCD is very similar to a paramagnetic substance. This explanation of asymptotic freedom is not new, but is apparently due to 't Hooft,[3] as quoted in[4] footnote 25.

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The argument is presented in Secs. II–IV, while technical details are relegated to four appendices: In Appendix A, the vacuum energy of a general field theory is derived. In Appendix B this is specialized to the vacuum energy of a charged field in an external magnetic field. In Appendix C, the connection to QCD is established. Finally, in Appendix D the familiar Euler summation formula is rederived for the sake of completeness.

We everywhere set $\hbar = c = 1$. The metric tensor used is $g_{\mu\nu} = (- - - +)$. Electrodynamic quantities and their counterpart in QCD are given in Heaviside–Lorentz rationalized units, which, e.g., means that potentials are related to the Gaussian units by a factor of $1/\sqrt{4\pi}$.

The calculational method used in our paper follows rather closely that of the pioneering paper of Landau,[5] which deals with the diamagnetic properties of a degenerate electron gas. Incomplete lists of references to original papers dealing with the calculation of the vacuum energy of a charged field in an external magnetic field are given in Ref. 6 for spins below 1 and in Refs. 7 and 8 for spin 1. Reference 7 contains in fact the earliest treatment of a phenomenon resembling asymptotic freedom recorded in the literature.

II. SCREENING VERSUS ANTISCREENING

We first recapitulate some properties of ordinary polarizable media.

In a polarizable medium, the potential energy $V(r)$ of two static test charges q_1 and q_2 is

$$V(r) = q_1 q_2 / 4\pi\epsilon r, \tag{2.1}$$

where r is the distance between the two charges. Here ϵ is the dielectric constant, which *in vacuo* takes the value 1. Ordinarily, the polarizability of the medium causes a screening of the interaction between the test charges, meaning

$$\epsilon > 1. \tag{2.2}$$

Antiscreening, on the other hand, corresponds to

$$\epsilon < 1. \tag{2.3}$$

A relativistic quantum field theory will have a vacuum that in many respects behaves like a polarizable medium. However, it differs from an ordinary polarizable medium on a very important point: it is relativistically invariant. This means, if we set the velocity of light $c = 1$, that the magnetic permeability μ is related to the dielectric constant ϵ by

$$\mu\epsilon = 1. \tag{2.4}$$

This consequence of Lorentz invariance in QCD is also important in recent theories on confinement of quarks and gluons through instantons.[9] Such a relationship will not exist for an ordinary polarizable material. (Thus a free-electron gas at zero temperature will exhibit both Pauli paramagnetism and Thomas–Fermi screening, so in this example both μ and ϵ are larger than unity.) Comparing (2.2) and (2.3) with (2.4) we learn that ordinary screening means

$$\mu < 1, \tag{2.5}$$

and antiscreening means

$$\mu > 1. \tag{2.6}$$

The magnetic permeability μ is usually written

$$\mu = 1 + 4\pi\chi, \tag{2.7}$$

where χ is the magnetic susceptibility, occurring in the expression giving the energy density of the medium E in the presence of an external magnetic field H:

$$E = -\tfrac{1}{2}4\pi\chi H^2. \tag{2.8}$$

If the medium contains magnetic dipoles, they tend to align themselves with the magnetic field, leading to a negative energy density, so in this case

$$\chi > 0 \tag{2.9}$$

and

$$\mu > 1, \tag{2.10}$$

and the material is called paramagnetic. Through the argument above, based upon Lorentz invariance that implied (2.4), we are led to associate this situation with antiscreening. On the other hand, the diamagnetic case where $\chi < 0$ is associated with screening.

Take the case of QED (quantum electrodynamics) described by the Lagrangian (in the notation of Ref. 10),

$$\mathscr{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma_{\gamma\mu}D^{\mu}\psi - m\bar{\psi}\psi, \tag{2.11}$$

where ψ is the electron field, A_μ the four-vector electromagnetic potential, and

$$D^{\mu}\psi = \partial^{\mu}\psi - ieA^{\mu}\psi, \tag{2.12}$$

$$F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}. \tag{2.13}$$

Two heavy static charges at distance r will here be surrounded by clouds of virtual electron–positron pairs ("virtual" means that they only occur in intermediate states of the quantum-mechanical perturbation theory). These clouds will be polarized by the test charges, producing a value of ϵ different from unity. But if the test charges are close together, they can penetrate each others' particle cloud and will thus not feel any screening or antiscreening effect. This means that, when adapting (2.1) to QED, we have to let $\epsilon \to 1$ for $r \to 0$, so the dielectric "constant" is in reality a function of r for this application.

Correcting (2.1) for the r dependence of ϵ,

$$V(r) = q_1 q_2 / 4\pi\epsilon(r) r, \tag{2.14}$$

we see that the effective strength of the interaction measured, e.g., by the "effective" charge of test charge 1,

$$q_1^2(r) = q_1^2/\epsilon(r), \tag{2.15}$$

in the screened case, where $\epsilon(r) > 1$, $\epsilon(r) \to 1$ for $r \to 0$, *increases* as $r \to 0$. This is what happens in QED. In the antiscreened case, on the other hand, where $\epsilon(r) < 1$, $\epsilon(r) \to 1$ for $r \to 0$, the effective strength of the interaction *decreases* as $r \to 0$. If it decreases to zero in the limit, we call the theory asymptotically free. This is in fact what happens in QCD, to which we now turn.

III. PERMEABILITY AND DIELECTRIC POLARIZATION OF THE QCD VACUUM

The Lagrangian density of QCD is

$$\mathscr{L} = -\tfrac{1}{4}G_{a\mu\nu}G_a^{\mu\nu} + \sum_F (i\bar{q}_F\gamma\partial_\mu D^\mu q_F - m_F\bar{q}_F q_F), \tag{3.1}$$

where

$$G_a^{\mu\nu} = \partial^\mu B_a^\nu - \partial^\nu B_a^\mu - g f_{abc} B_b^\mu B_c^\nu, \tag{3.2}$$

$$D^\mu q_F = \partial^\mu q^F + \tfrac{1}{2} i g B_a^\mu \lambda_a q, \tag{3.3}$$

with q_F as the quark field, transforming according to the fundamental representation of $SU(3)$, with structure constants f_{abc} and representation matrices λ_a, whereas B^μ_a is the gluon field. The gauge group $SU(3)$ is called the color group. The quarks have, besides the color degree of freedom, an additional internal degree of freedom called flavor, indicated in (3.1) by the label F; the quark mass m_F is flavor dependent.

Flavor quantum numbers were first introduced for the description of nucleons, where the proton was associated with isospin up and the neutron with isospin down. Later on strange particles were discovered, necessitating a new flavor quantum number: strangeness. There are now five experimentally established flavors: up, down, strangeness, charm, and bottom.

The gluon part of (2.11) contains both a kinetic term,

$$\mathscr{L}_{\text{kin,gluon}} = -\tfrac{1}{4}(\partial_\mu B_{a\nu} - \partial_\nu B_{a\mu})(\partial^\mu B^\nu_a - \partial^\nu B^\mu_a), \quad (3.4)$$

and an interaction term,

$$\mathscr{L}_{\text{int,gluon}} = \tfrac{1}{2}g f_{abc}(\partial_\mu B_{a\nu} - \partial_\nu B_{a\mu}) B^\mu_b B^\nu_c - \tfrac{1}{4}g^2 f_{abc} f_{ab'c'} B_{b\mu} B_{c\nu} B^\mu_{b'} B^\nu_{c'}. \quad (3.5)$$

The form of the kinetic term is the same form as the photon term of the QED Lagrangian, i.e., the first term of (2.11). Thus exchange of gluons gives rise to forces similar to the Coulomb interaction, but acting on particles with color instead of charge. One example of a particle with color is the quark, as is apparent from (3.1) and (3.3). But (3.5) shows that the gluons carry color themselves (unlike photons that don't carry any charge), and this is the circumstance that makes QCD asymptotically free.

The arguments on vacuum polarization given in Sec. II apply also to QCD. We can here let two heavy colored test particles interact and will again get an interaction potential of the form (2.1). The properties of the dielectric constant ϵ are also here due to clouds of virtual particles, but there will also be gluons in these clouds because of the gluon self-coupling.

The gluons are particles with spin. Thus the effect of virtual gluons on the vacuum of the theory is likely to make it behave like a paramagnetic medium. According to the arguments presented in Sec. II, this implies antiscreening.

At this point one might worry about ordinary QED with the Lagrangian density (2.11). Vacuum of this theory contains virtual pairs of electrons and positrons that have spin. Thus according to the argument given above, it should behave like a paramagnetic substance, and this in its turn would imply antiscreening and asymptotic freedom. It is, however, a consequence of the calculations in Appendices A and B that the QED vacuum has normal diamagnetic properties, and this can in the present context be considered a consequence of the Pauli exclusion principle. The exclusion principle does not apply to gluons, which in contrast to electrons have integer spin, and so the paramagnetic vacuum becomes possible in QCD.

An explicit calculation of the vacuum energy of QCD in an external homogeneous constant color magnetic field H, carried out in Appendices A, B, and C, indeed confirms that the vacuum is paramagnetic. The final result is given in (C13) and is

$$E_{\text{vac,QCD}} = -\tfrac{1}{2}VH^2 \frac{(33 - 2N_F)g^2}{48\pi^2} \log \frac{\Lambda^2}{|gH|}, \quad (3.6)$$

where an insignificant term was disregarded. Here paramagnetism manifests itself through the minus sign in front of the right-hand side, which shows that the vacuum energy is decreased by application of an external color magnetic field. The unexplained symbols mean: N_F the number of quark species (flavors), V the quantization volume, and Λ a cutoff introduced in such a way that only the contribution of field modes with energy less than Λ has been taken into account in (3.6). It is shown in Appendices B and C how the coefficient $33 - 2N_F$ in (3.6) can be understood in a simple way from Euler's summation formula (D1):

$$\sum_{n=0}^{\infty} f(n + \tfrac{1}{2}) = \int_0^\infty f(x)dx - \frac{1}{24} f'(x)\Big|_0^\infty + \cdots. \quad (3.7)$$

The appearance of a cutoff Λ in intermediate calculations is a standard problem in quantum field theory. It is usually resolved by doing a so-called renormalization. In the present context we need not be concerned about it if we accept that the theory is mutilated at too high energies or too short distances.

Comparison of (3.6) and (2.8) shows that the appearance of a logarithm in (3.6) prevents us from introducing a magnetic susceptibility χ. What we can do is to introduce an "effective" susceptibility $\chi(H)$ such that

$$E_{\text{vac,QCD}} = -\tfrac{1}{2}4\pi\chi(H)H^2; \quad (3.8)$$

and an "effective" permeability by

$$\mu(H) = 1 + 4\pi\chi(H), \quad (3.9)$$

which according to (3.6) is

$$\mu(H) \simeq 1 + \frac{(33 - 2N_F)g^2}{48\pi^2} \log \frac{\Lambda^2}{|gH|}. \quad (3.10)$$

The relationship (2.4) between the permeability μ and the dielectric constant ϵ can obviously no longer be maintained in unchanged form. However, already dimensional reasons make it plausible that the "effective" permeability $\mu(H)$ and the "effective" dielectric constant $\epsilon(r)$ should be related by

$$\epsilon(r) = \mu(H)^{-1}\big|_{|gH| \to 1/r^2}. \quad (3.11)$$

The correctness of this relation is supported further by a heuristic argument given in the following paragraph. A rigorous proof involves use of the so-called renormalization group.

The value of $\mu(H)$ given in (3.10) was obtained by taking into account gluon and quark field modes with wavelength λ restricted by

$$1/\sqrt{|gH|} \gtrsim \lambda \gtrsim 1/\Lambda. \quad (3.12)$$

The upper bound $1/\sqrt{|gH|}$ on λ is due to the fact that the eigenstates of the QCD Hamiltonian in an external magnetic field, as seen from the calculations in Appendix B, are described in one direction by harmonic oscillator wave functions with extension of the order $1/\sqrt{|gH|}$. The effective dielectric constant $\epsilon(r)$ might be calculated by computing the vacuum energy of the virtual gluons producing the static potential between a pair of heavy colored test particles at distance r from each other. The wavelength λ of the field modes to be summed should in this case be restricted by

$$r \gtrsim \lambda \gtrsim 1/\Lambda. \quad (3.13)$$

To include field modes with wavelengths in the same interval when calculating $\mu(H)$ and $\epsilon(r)$, respectively, we have to choose

$$r \simeq 1/\sqrt{|gH|}. \quad (3.14)$$

It is obvious from this argument that the correspondence between r and $1/\sqrt{|gH|}$ is not a very precise one, but since we are only interested in the logarithmic terms of $\mu(H)$ and $\epsilon(r)$ anyway, it is in fact sufficient for our purposes.

Combining (3.10) and (3.11), we learn

$$\epsilon(r) \simeq \left(1 + \frac{(33 - 2N_F)g^2}{24\pi^2} \log(\Lambda r)\right)^{-1}. \tag{3.15}$$

This function is 1 for $r = 1/\Lambda$ instead of at $r = 0$. This we interpret as meaning that in the presence of an energy cutoff Λ there is a minimal length in the theory $1/\Lambda$ (we work in units where $\hbar = c = 1$; otherwise the minimal length would be $\hbar c/\Lambda$). Only for $\Lambda \to \infty$ is it meaningful to consider the case $r = 0$ in the context of the theory.

At distances larger than $r = 1/\Lambda$ the function $\epsilon(r)$ decreases, producing antiscreening.

IV. ASYMPTOTIC FREEDOM

We can now test for asymptotic freedom. We must consider an "effective" coupling strength of one of the static colored test particles similar to (2.15):

$$q^2(r) = q^2/\epsilon(r) \tag{4.1}$$

in the limit where $r \to 0$. We consider the ratio between $q^2(r)$ at the minimal distance $r_1 = 1/\Lambda$ and at a fixed distance r_2:

$$\frac{q^2(r_1)}{q^2(r_2)} = \frac{\epsilon(r_2)}{\epsilon(r_1)} = \left(1 + \frac{(33 - 2N_F)g^2}{24\pi^2} \log \frac{r_2}{r_1}\right)^{-1}, \tag{4.2}$$

whence

$$\frac{q^2(r_1)}{q^2(r_2)} \to 0 \quad \text{for } r_1 \to 0, \tag{4.3}$$

provided $N_F \leqslant 16$, so the effective strength of the interaction indeed decreases to zero in the short-distance limit, and the theory is asymptotically free.

Usually the property of asymptotic freedom is expressed in terms of the so-called β function, which in the present context can be defined by

$$\beta(g) = -\frac{g}{2} r \frac{\partial}{\partial r} \frac{q^2(r)}{q^2}. \tag{4.4}$$

From (4.1) and (3.15) we immediately get

$$\beta(g) = -\frac{g}{2} r \frac{\partial}{\partial r} \frac{1}{\epsilon(r)} = -\frac{(33 - 2N_F)g^3}{48\pi^2}. \tag{4.5}$$

Asymptotic freedom here manifests itself through the minus sign in front of the last form of (4.5) (assuming $N_F \leqslant 16$). This one can verify by integrating (4.4) to obtain $q^2(r)$ expressed through $\beta(g)$; a positive coefficient of g^3 in the last form of (4.5) would clearly upset asymptotic freedom.

V. CONCLUSION

We have calculated the effective coupling strength and the β function for QCD by treating vacuum as a polarizable medium in a relatively intuitive way.

The main ingredients in the argument were (a) Lorentz invariance, which relates the dielectric constant and the magnetic permeability; and (b) the Euler summation formula, making a simple evaluation of the vacuum energy in an external constant homogeneous magnetic field possible.

It was found that the qualitative aspects of asymptotic freedom could be related to the QCD vacuum behaving like a paramagnetic substance. On the quantitative level, both the coefficients of the vacuum energy and of the β function could be understood through the factor

$$-(-1)^{2s} \sum_{s_3} (s_3^2/2 - 1/24)$$

arising through the calculations of Appendix B [see (B.24) and earlier formulas], where s is the total spin of the field in question, and s_3 the spin projection on the direction of the external magnetic field.

Our conclusion is that the asymptotic freedom of QCD both on the qualitative and the quantitative level can be considered an effect of the gluon spin (with a small quantitative correction from the quark spin).

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APPENDIX A: VACUUM ENERGIES

A quantity of fundamental importance in quantum field theories is the vacuum energy. We now first determine this quantity in the very simplest case, that of a free complex massless scalar field ϕ.

The Lagrangian density is

$$\mathscr{L} = g_{\mu\nu} \partial^\mu \phi^+ \partial^\nu \phi, \tag{A1}$$

whence the canonical momenta are found:

$$\pi = \frac{\partial \mathscr{L}}{\partial \partial_0 \phi} = \partial_0 \phi^+, \tag{A2a}$$

$$\pi^+ = \partial_0 \phi, \tag{A2b}$$

as well as the Hamiltonian density

$$\mathscr{H} = \pi \partial_0 \phi + \pi^+ \partial_0 \phi^+ - \mathscr{L} = \pi^+ \pi + \nabla \phi^+ \nabla \phi. \tag{A3}$$

The canonical commutation relations in the Heisenberg picture are

$$[\pi(\mathbf{x},t), \phi(\mathbf{x}',t)] = [\pi^+(\mathbf{x},t), \phi^+(\mathbf{x}',t)] = -i\delta(\mathbf{x} - \mathbf{x}') \tag{A4}$$

with all other equal-time commutators involving fields and canonical momenta vanishing. We perform a Fourier transformation of ϕ and ϕ^+:

$$\phi(\mathbf{x},t) = \sum_{\mathbf{k}} [\phi_{\mathbf{k}}(\mathbf{x}) e^{-iE_{\mathbf{k}}t} a_{\mathbf{k}} + {}^*_{\mathbf{k}}(\mathbf{x}) e^{iE_{\mathbf{k}}t} b^+_{\mathbf{k}}], \tag{A5a}$$

$$\phi^+(\mathbf{x},t) = \sum_{\mathbf{k}} \{\phi_{\mathbf{k}}(\mathbf{x}) e^{-iE_{\mathbf{k}}t} b_{\mathbf{k}} + \phi^*_{\mathbf{k}}(\mathbf{x}) e^{iE_{\mathbf{k}}t} a^+_{\mathbf{k}}\}, \tag{A5b}$$

where

$$\phi_{\mathbf{k}}(\mathbf{x}) = [1/\surd(2E_{\mathbf{k}} V)] e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{A6}$$

$$E_{\mathbf{k}} = \surd(\mathbf{k}^2), \tag{A7}$$

and V is the quantization volume.

The canonical commutation relations imply for the new operators a and b:

$$[a_{\mathbf{k}}, a^+_{\mathbf{k}'}] = [b_{\mathbf{k}}, b^+_{\mathbf{k}'}] = \delta_{\mathbf{k}\cdot\mathbf{k}'}; \tag{A8}$$

and all other commutators of a and b operators are zero. Here

$$\delta_{\mathbf{k}\cdot\mathbf{k}'} = \delta_{k_1 k_1'} \delta_{k_2 k_2'} \delta_{k_3 k_3'}$$

is Kronecker's δ function.

From (A8) one gets the standard interpretation of the a

and b operators: $a_{\mathbf{k}}^{+}$ and $b_{\mathbf{k}}^{+}$ create a particle and an antiparticle, respectively, both having momentum **k**, whereas $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$ are destruction operators for a particle and an antiparticle, also with momentum **k**. The vacuum state Ω_0, where neither particles nor antiparticles are present, has the property

$$a_{\mathbf{k}}\Omega_0 = b_{\mathbf{k}}\Omega_0 = 0. \tag{A9}$$

Inserting (A5) into (A3) and using the orthonormality of the $\phi_{\mathbf{k}}$ functions we next determine the Hamiltonian H:

$$H \equiv \int d^3\mathbf{x}\, \mathscr{H}(\mathbf{x},t) = \sum_{\mathbf{k}} E_{\mathbf{k}}(a_{\mathbf{k}}^{+}a_{\mathbf{k}} + b_{\mathbf{k}}b_{\mathbf{k}}^{+}). \tag{A10}$$

It is here more convenient to use a normal ordering where all annihilation operators are moved to the right. Doing this by (A8) we get

$$H = \sum_{\mathbf{k}} E_{\mathbf{k}}(a_{\mathbf{k}}^{+}a_{\mathbf{k}} + b_{\mathbf{k}}^{+}b_{\mathbf{k}} + 1). \tag{A11}$$

From (A9) and (A11) we can now easily find the vacuum energy, which is the vacuum-expectation value of the Hamiltonian:

$$E_{\text{vac}} = (\Omega_0, H\Omega_0) = \sum_{\mathbf{k}} E_{\mathbf{k}}. \tag{A12}$$

E_{vac} is obviously closely related to the zero-point energy of a quantized harmonic oscillator.

The vacuum energy is not very interesting for a free field. It is not even meaningful since it is badly divergent. But the considerations carried through here also apply to the situation where vacuum is changed by some external agent, e.g., a magnetic field. In that case the most divergent part of E_{vac} can be isolated and the properties of the rest studied, as will be shown in detail in Appendix B. Furthermore, the argument also applies in essentially unchanged form to fields with spin, if the spin is integral.

This excludes the spin-$\frac{1}{2}$ field, and the way to calculate the vacuum energy in this case is now briefly described.

In the conventions of Bjorken and Drell,[9] the Langrangian density of a massless Dirac spin-$\frac{1}{2}$ field is

$$\mathscr{L} = i\bar{\psi}\partial\gamma_\mu\partial^\mu\psi, \tag{A13}$$

and the Hamiltonian density is

$$\mathscr{H} = -i\bar{\psi}\gamma\nabla\psi. \tag{A14}$$

Corresponding to (A5) we have here the decomposition

$$\psi(x) = \sum_{\mathbf{k},\sigma}\left[u_{\mathbf{k},\sigma}(\mathbf{x})e^{-iE_{\mathbf{k}}t}a_{\mathbf{k},\sigma} + v_{\mathbf{k},\sigma}(\mathbf{x})e^{iE_{\mathbf{k}}t}b_{\mathbf{k},\sigma}^{+}\right], \tag{A15}$$

where $u_{\mathbf{k},\sigma}(\mathbf{x})$ and $v_{\mathbf{k},\sigma}(\mathbf{x})$ are four-component spinors with σ denoting the spin projection on the direction of **k**. Again $a_{\mathbf{k},\sigma}$ is the annihilation operator of a particle and $b_{\mathbf{k},\sigma}^{+}$ the creation operator of an antiparticle, but the commutation relations are now, because of the Pauli exclusion principle, replaced by anticommutation relations

$$(a_{\mathbf{k},\sigma}, a_{\mathbf{k}',\sigma'}^{+}) = (b_{\mathbf{k},\sigma}, b_{\mathbf{k}',\sigma'}^{+}) = \delta_{\mathbf{k}\cdot\mathbf{k}'}\delta_{\sigma'}, \tag{A16}$$

etc.

The Hamiltonian here is

$$H \equiv \int d^3\mathbf{x}\, \mathscr{H}(\mathbf{x},t) = \sum_{\mathbf{k},\sigma} E_{\mathbf{k}}(a_{\mathbf{k},\sigma}^{+}a_{\mathbf{k},\sigma} - b_{\mathbf{k},\sigma}b_{\mathbf{k},\sigma}^{+}), \tag{A17}$$

which by (A16) also is

$$H = \sum_{\mathbf{k},\sigma} E_{\mathbf{k}}(a_{\mathbf{k},\sigma}^{+}a_{\mathbf{k},\sigma} + b_{\mathbf{k},\sigma}^{+}b_{\mathbf{k},\sigma} - 1), \tag{A18}$$

so the vacuum energy becomes

$$E_{\text{vac}} = -\sum_{\mathbf{k},\sigma} E_{\mathbf{k}} = -2\sum_{\mathbf{k}} E_{\mathbf{k}}. \tag{A19}$$

Remarkably enough, there is a sign difference relative to the vacuum energy in the integer-spin case.

APPENDIX B: VACUUM ENERGIES IN THE PRESENCE OF AN EXTERNAL MAGNETIC FIELD

We now turn to the determination of vacuum energies for field theories in the presence of an external magnetic field. Choosing the magnetic field H along the 3-direction in space, a possible vector potential A_μ has components

$$A_2 = Hx_1, A_1 = A_3 = A_0 = 0.$$

The simplest example is the charged scalar field ϕ for which the Lagrangian density instead of (A.1) becomes

$$\mathscr{L} = g_{\mu\nu}D^{\mu *}\phi^{+}D^{\nu}\phi, \tag{B1}$$

$$D^\mu = \partial^\mu - ieA^\mu \tag{B2}$$

(we keep the field massless).

Constructing the Hamiltonian and inserting the vector potential chosen above, we find that the energy values are determined by the eigenvalue equation

$$\left(\Delta - e^2H^2x_1^2 - 2ieHx_1\frac{\partial}{\partial x_2}\right)\phi_E = -E^2\phi_E. \tag{B3}$$

The solutions of this equation have the form

$$\phi_E(x_1,x_2,x_3) = e^{i(k_2x_2 + k_3x_3)}\phi_n(x_1 - k_2/eH), \tag{B4}$$

where ϕ_n fulfills

$$\left(\frac{\partial^2}{\partial x^2} - e^2H^2x^2\right)\phi_n(x) = -(E^2 - k_3^2)\phi_n(x). \tag{B5}$$

We immediately recognize a harmonic oscillator problem with frequency $|eH|$, so

$$E^2 = k_3^2 + 2|eH|(n + \tfrac{1}{2}) \quad n = 0,1,2,\ldots . \tag{B6}$$

For spin $\frac{1}{2}$ and spin 1 the eigenvalue equations are simple generalizations of (B3), viz.,

$$\left(\Delta - e^2H^2x_1^2 - 2ieHx_1\frac{\partial}{\partial x_2} + 2eHS_3\right)\phi_E = -e^2\phi_E \tag{B7}$$

for specific choices of the field Lagrangians. Here S_3 is the projection of the spin matrix on the direction of the magnetic field. For spin $\frac{1}{2}$ this is true for the ordinary Dirac Lagrangian:

$$\mathscr{L} = i\bar{\psi}\gamma_\mu D^\mu\psi, \tag{B8}$$

while for spin 1 the Lagrangian has to be chosen as

$$\mathscr{L} = -\tfrac{1}{2}(D_\mu^* W_\nu^{+} - D_\nu^* W_\mu^{+}) \times (D^\mu W^\nu - D^\nu W^\mu) - ieF^{\mu\nu}W_\mu^{+}W_\nu, \tag{B9}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength. Furthermore the vector field W_μ is subject to the gauge condition

$$D_\mu W^\mu = 0, \tag{B10}$$

which eliminates unphysical degrees of freedom.

The solution of the eigenvalue equation (B7) is

$$E^2 = 2|eH|(n + \tfrac{1}{2}) + k_3^2 - 2eHs_3, \tag{B11}$$

where s_3 is an eigenvalue of the matrix S_3. We can then write the vacuum energy:

$$E_{vac} = (-1)^{2s}\frac{V|eH|}{4\pi^2} \times \int dk_3 \sum_{n=0}^{\infty}\sum_{s_3} \sqrt{[2|eH|(n+\tfrac{1}{2}) + k_3^2 - 2eHs_3]}. \quad (B12)$$

Two points in this formula need explanation. The first is the sign factor. s denotes the total spin, which here may take the values 0, $\frac{1}{2}$, and 1. As explained above we have to use an overall minus sign in the vacuum energy for spin $\frac{1}{2}$. The second point is the factor $V|eH|/4\pi^2$ giving the density of states. To verify this, we enclose the system in a box with the extension L_1, L_2, and L_3 in the one, two, and three directions, respectively, so $V = L_1L_2L_3$. The number of possible values of k_2 and k_3 in the intervals Δk_2 and Δk_3 are then given by the usual values

$$\Delta n_2 = (L_2/2\pi)\Delta k_2, \quad (B13a)$$

$$\Delta n_3 = (L_3/2\pi)\Delta k_3. \quad (B13b)$$

In the 1 direction we see from (B4) that the eigenfunctions are those of a displaced harmonic oscillator with equilibrium position at $x_1 = k_2/eH$. Obviously we have to require

$$0 \leqslant k_2/|eH| \leqslant L_1, \quad (B14)$$

i.e., from (B13a),

$$n_2 = (L_1L_2/2\pi)|eH|, \quad (B15)$$

so for fixed values of the oscillator quantum number n and of s_3 we have

$$\frac{L_1L_2L_3|eH|}{4\pi^2}\Delta k_3 = \frac{V|eH|}{4\pi^2}\Delta k_3 \quad (B16)$$

states with values of k_3 in the interval Δk_3.

Equation (B12) can be dealt with by the Euler summation formula (D1). However, we have to be careful, because the expression as it stands is divergent. To handle this difficulty, we restrict the sum to run over terms not larger than a cutoff Λ^2:

$$E_{vac} = (-1)^{2s}\frac{V|eH|}{4\pi^2}\int dk_3 \times \sum_{n=0}^{\infty}\sum_{s_3}\sqrt{[2|eH|(n+\tfrac{1}{2}) + k_3^2 - 2eHs_3]} \times \theta(\Lambda^2 - 2|eH|(n+\tfrac{1}{2}) - k_3^2 + 2eHs_3)$$
$$= (-1)^{2s}\sum_{n=0}^{\infty}\sum_{s_3} f\left(n + \tfrac{1}{2} - \frac{eH}{|eH|}S_3\right), \quad (B12')$$

where

$$f(x) \equiv \frac{V|eH|}{4\pi^2}\int dk_3 \sqrt{(2|eH|x + k_3^2)}\,\theta(\Lambda^2 - 2|eH|x - k_3^2)$$

$$= \frac{V|eH|}{4\pi^2}\left[2|eH|x \log\left(\sqrt{\frac{\Lambda^2}{2|eH|x}} + \sqrt{\frac{\Lambda^2}{2|eH|x} - 1}\right) + \sqrt{\Lambda^2(\Lambda^2 - 2|eH|x)}\,\right]. \quad (B17)$$

In order to apply (D1) to the sum in (B12′) we have to restrict the sum to values of n where f only varies slowly. We do this by splitting the sum:

$$E_{vac} = (-1)^{2s}\sum_{n=N}^{\infty}\sum_{s_3} f[n + \tfrac{1}{2} - (eH/|eH|)S_3] + \Phi(N,|eH|), \quad (B18a)$$

$$\Phi(N,|eH| = (-1)^{2s}\sum_{n=0}^{N-1}\sum_{s_3} f[n + \tfrac{1}{2} - (eH/|eH|)S_3], \quad (B18b)$$

where N is an integer large enough to make the Euler summation formula applicable for $n > N$, i.e.,

$$\left|\frac{f[n+1+\tfrac{1}{2} - (eH/|eH|)S_3] - f[n+\tfrac{1}{2} - (eH/|eH|)S_3]}{f[n+\tfrac{1}{2} - (eH|eH|)S_3]}\right| \ll 1. \quad (B19)$$

On the other hand, we require that N be chosen relative to the cutoff Λ in such a way that

$$N \ll \Lambda^2/2|eH|. \quad (B20)$$

Applying now the Euler summation formula (D1) to the first term on the right-hand side of (B18a), we get

$$E_{vac} \simeq (-1)^{2s}\sum_{s_3}\left[\int_N^{\infty} f\left(x - \frac{eH}{|eH|}S_3\right)dx - \tfrac{1}{24}f'\left(x - \frac{eH}{|eH|}S_3\right)\Big|_0^{\infty}\right] + \Phi(N,|eH|)$$
$$\simeq (-1)^{2s}\sum_{s_3}\left[\int_N^{\infty} f(x)dx - \left(\frac{s_3^2}{2} - \frac{1}{24}\right)f'(N)\right] + \Phi(N,|eH|),$$

where the final form of the equation was obtained through a Taylor series expansion. The value of $f'(N)$ can be found from (B17):

$$f'(N) \simeq \frac{V|eH|}{4\pi^2}\left(|eH|\log\frac{2\Lambda^2}{|eH|N} - 2|eH|\right). \quad (B22)$$

Here it is convenient to rearrange (B21):

$$E_{vac} \simeq (-1)^{2s}\sum_{s_3}\left[\int_0^{\infty} f(x)dx - \left(\frac{s_3^2}{2} - \frac{1}{24}\right) \times \frac{Ve^2H^2}{4\pi^2}\log\frac{\Lambda^2}{|eH|}\right] + \tilde{\Phi}, \quad (B23a)$$

$$\tilde{\Phi} = \Phi(N,|eH|) - (-1)^{2s}\sum_{s_3}\left[\int_0^N f(x)dx - \left(\frac{s_3^2}{2} - \frac{1}{24}\right)\frac{Ve^2H^2}{4\pi^2}\left(\log\frac{2}{N} - 2\right)\right]. \quad (B23b)$$

The quantity $\tilde{\Phi}$ does not depend upon N because E_{vac} does not, and by inspection it is easily found that $\tilde{\Phi}$ also is independent of the cutoff Λ. Then from dimensional reasons it follows that $\tilde{\Phi}$ has the form e^2H^2 times a constant. Also the first term in (B23a),

$$(-1)^{2s}\sum_{s_3}\int_0^{\infty} f(x)dx,$$

is independent of the external magnetic field H. It can be identified with the vacuum energy in the absence of any external agent, and we get rid of it by a redefinition of $E_{\rm vac}$:

$$E_{\rm vac}\to E_{\rm vac}-(-1)^{2s}\sum_{s_3}\int_0^\infty f(x)dx.$$

The value of the redefined vacuum energy is thus

$$E_{\rm vac}=-(-1)^{2s}\sum_{s_3}\left(\frac{s_3^2}{2}-\frac{1}{24}\right)\frac{Ve^2H^2}{4\pi^2}\log\frac{\Lambda^2}{|eH|}+\tilde{\Phi}, \tag{B24}$$

and here $\tilde{\Phi}$ can actually be neglected for the purposes we have in mind.

APPENDIX C: CONNECTION TO QCD

It is shown here first how to identify components of the gluon field B_a^μ with a photonlike vector field as well as charged vector fields. This identification makes it possible, from the results of Appendix B, to determine the vacuum energy of the gluon field in the presence of an external magnetic color field.

Take first the case of the gauge group $SU(2)$. In this case the gluonic Lagrangian is

$$\mathscr{L}=-\tfrac{1}{4}G_{a\mu\nu}G_a^{\mu\nu}, \tag{C1a}$$

$$G_a^{\mu\nu}=\partial^\mu B_a^\nu-\partial^\nu B_a^\mu-g\epsilon_{abc}B_b^\mu B_c^\nu, \tag{C1b}$$

where ϵ_{abc} is the usual antisymmetric symbol with three indices:

$$\epsilon_{abc}=\begin{cases}1 & \text{for } abc \text{ an even permutation of } 1,2,3\\ -1 & \text{for } abc \text{ an odd permutation of } 1,2,3\\ 0 & \text{otherwise.}\end{cases} \tag{C2}$$

We now set

$$B_3^\mu\equiv A^\mu, \tag{C3a}$$

$$B_1^\mu\equiv(1/\sqrt{2})(W^\mu+W^{+\mu}), \tag{C3b}$$

$$B_2^\mu\equiv(1/i\sqrt{2})(W^\mu-W^{+\mu}). \tag{C3c}$$

Expressed in terms of the new field variables, the Lagrangian (C1a) becomes

$$\mathscr{L}=-\tfrac{1}{2}(D_\mu W_\nu-D_\nu W_\mu)(D^{\mu*}W^{\nu+}-D^{\nu*}W^{\mu+})-\tfrac{1}{4}[\partial_\mu A_\nu-\partial_\nu A_\mu+ig(W_\mu^+W_\nu-W_\mu W_\nu^+)]\times[\partial^\mu A^\nu-\partial^\nu A^\mu+ig(W^{\mu+}W^\nu-W^\mu W^{\nu+})], \tag{C4a}$$

$$D^\mu=\partial^\mu-igA^\mu. \tag{C4b}$$

A comparison of (C4a) with (B9a) shows that the part of (C4a) having a linear dependence of both W and W^+ is identical to (B9a), provided we substitute for the gluon coupling constant g the charge e, and we identify the field A_μ introduced in (C3a) with the electromagnetic vector potential. Thus having an external gluon field of the form

$$B_1^\mu=B_2^\mu=0,\quad B_3^\mu=0\quad \text{for}\ \mu\neq 2,\quad B_3^2=Hx^1,$$

we can immediately use the result (B24) to compute the vacuum energy. Restricting the sum over s_3 to ± 1, since these are the only possible values for a massless vector particle, we obtain

$$E_{\rm vac}\simeq-\frac{11Vg^2H^2}{48\pi^2}\log\frac{\Lambda^2}{|gH|}, \tag{C5}$$

where the nonlogarithmic term was left out.

Next we consider the case of physical interest, where the gauge group is $SU(3)$. In this case there are eight gluon fields, and it is convenient to use

$$B_8^\mu=A^\mu, \tag{C6}$$

and the nonvanishing component of the external gluon field is now $B_8^2=Hx^1$. A list of the structure constants can be found, e.g., in Ref. 11. We need the following values:

$$f_{458}=f_{678}=\sqrt{3}/2. \tag{C7}$$

The structure constants are again completely antisymmetric in their indices. We are here led to introduce two W fields:

$$W_{1\mu}\equiv(1/\sqrt{2})(B_{4\mu}+iB_{5\mu}), \tag{C8a}$$

$$W_{2\mu}\equiv(1/\sqrt{2})(B_{6\mu}+iB_{7\mu}), \tag{C8b}$$

and the part of the gluon Lagrangian that either is linear in both W_1^μ and $W_1^{+\mu}$ or W_2^μ and $W_2^{+\mu}$ is

$$\mathscr{L}=-\tfrac{1}{2}\sum_{i=1}^{2}(D_\mu W_{i\nu}-D_\nu W_{i\mu})(D^{\mu*}W_i^{\nu+}-D^{\nu*}W_i^{\mu+})-ig(\sqrt{3}/2)(\partial^\mu A^\nu-\partial^\nu A^\mu)\sum_{i=1}^{2}W_{i\mu}^+W_{i\nu}, \tag{C9a}$$

where now

$$D^\mu=\partial^\mu-ig\sqrt{3}/2. \tag{C9b}$$

The effect of the new values of the structure constants relative to the $SU(2)$ case is obviously that g has to be replaced by $(\sqrt{3}/2)g$, and furthermore the vacuum energy gets a factor 2, since there are now two W fields, so discarding again nonlogarithmic terms we get

$$E_{\rm vac,gluon}\simeq-\frac{33Vg^2H^2}{96\pi^2}\log\frac{\Lambda^2}{|gH|}. \tag{C10}$$

In order to obtain the complete vacuum energy for QCD in the external gluon field configuration considered here we have to determine also the contribution of the quarks. We see from (3.1) and (3.3) that we only need the matrix λ_8, which according to Ref. 11, p. 5 is

$$\lambda_8=\begin{Bmatrix}\frac{1}{\sqrt{3}} & 0 & 0\\ 0 & \frac{1}{\sqrt{3}} & 0\\ 0 & 0 & -\frac{2}{\sqrt{3}}\end{Bmatrix}. \tag{C11}$$

The vacuum energy of the quarks is thus equal to that of three charged spin-$\frac{1}{2}$ particles with charges $-g/2\sqrt{3}$, $-g/2\sqrt{3}$, and $g/\sqrt{3}$, respectively, i.e., by (B24):

$$\frac{Vg^2H^2}{48\pi^2}\log\frac{\Lambda^2}{|gH|}. \tag{C12}$$

So the total vacuum energy of QCD is, in the approximation where only logarithmic terms are kept,

$$E_{\rm vac,QCD}\simeq-\frac{(33-2N_F)Vg^2H^2}{96\pi^2}\log\frac{\Lambda^2}{|gH|}, \tag{C13}$$

where N_F is the number of species (flavors) of quarks.

Here we have neglected the influence of the quark masses. Going back to Appendix B, we see that the effect of the quark masses is to make the function $\tilde{\Phi}$ introduced in (B23b) depend on the ratios between quark masses and $|gH|$. This new functional dependence on $\sqrt{|gH|}$ might in principle upset the results valid for massless quarks; however, it is rather easy to verify that this is not the case, so the quark masses may safely be neglected in the arguments of Secs. II–IV leading to asymptotic freedom of QCD.

APPENDIX D: EULER SUMMATION FORMULA

The Euler summation formula is

$$\sum_{\nu=n_1}^{n_2-1} f(\varphi + \tfrac{1}{2}) = \int_{n_1}^{n_2} f(x)dx - \frac{1}{24} f'(x)\Big|_{n_1}^{n_2} + \cdots. \qquad \text{(D1)}$$

In order to prove it we consider

$$\int_n^{n+1} f(x)dx - f(n + \tfrac{1}{2})$$

$$= \int_n^{n+1} \sum_{t=1}^{\infty} \frac{(x-n-\frac{1}{2})^t}{t!} f^{(t)}(n+\tfrac{1}{2})dx$$

$$= \int_n^{n+1} \frac{(x-n-\frac{1}{2})^2}{2} f''(n+\tfrac{1}{2})dx + \cdots = \frac{1}{24} f''(n+\tfrac{1}{2}) + \cdots,$$

i.e.,

$$\sum_{\nu=n_1}^{n_2-1} f(\nu + \tfrac{1}{2}) = \int_{n_1}^{n_2} f(x)dx - \frac{1}{24} \sum_{\nu=n_1}^{n_2-1} f''(\nu + \tfrac{1}{2}) + \cdots.$$

Approximating now the sum in the second term on the right-hand side by an integral, the summation formula follows. It obviously only provides a useful approximation, if the function varies slowly.

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[1]G.'t Hooft, Marseille meeting on field theory 1972 (unpublished); and Nucl. Phys. B 62, 444 (1973); H. D. Politzer, Phys. Rev. Lett. 30, 1346 (1973); D. J. Gross and F. Wilczek, *ibid.* 30, 1343 (1973).
[2]Ernest Ma, Am. J. Phys. 47, 873 (1979).
[3]G. 't Hooft (unpublished).
[4]P. D. Mannheim, Phys. Rev. D 9, 3438 (1974).
[5]L. D. Landau, Z. Phys. 64, 629 (1931).
[6]W. Heisenberg and H. Euler, Z. Phys. 98, 714 (1936); V. F. Weisskopf, Mat. Fys. Medd. Dan. Vid. Selsk. 14(6), 3 (1936). J. Schwinger, Phys. Rev. 82, 664 (1951).
[7]V. S. Vanyashin and M. V. Terenteev, Sov. Phys. JETP 21, 375 (1965).
[8]A. Salam and J. Strathdee, Nucl. Phys. B 90, 203 (1975); M. J. Duff and R. Ramón-Medrano, Phys. Rev. D 12, 3357 (1976); V. V. Skalozub, Sov. J. Nucl. Phys. 21, 690 (1976); G. K. Savvidy, Phys. Lett. 71B, 133 (1977); S. G. Matinyan and G. K. Savvidy, Nucl. Phys. B 134, 539 (1978); N. K. Nielsen and P. Olesen, *ibid.* 144, 376 (1978).
[9]C. G. Callan, R. F. Dashen, and D. J. Gross, Phys. Rev. D 19, 1826, (1979).
[10]J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).
[11]S. L. Adler and R. F. Dashen, *Current Algebras* (Benjamin, New York and Amsterdam, 1968).

Measurement of the atmospheric electrostatic potential gradient near sea level

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A simple apparatus has been constructed to measure the atmospheric electrostatic-potential gradient at the surface of the Earth. On 10 September 1979 in an open field in South Hadley, Massachusetts (elevation 250 ft) a potential gradient of 197 V/m was obtained, in reasonable agreement with previous measurements in comparable locations and under similar weather conditions.

INTRODUCTION

It is well known that there is a potential difference of 400 000 V between the ionosphere and the surface of the Earth.[1] This gives rise to a positive ion current directed toward the Earth and produces an average electrostatic potential gradient near the surface of the Earth of 100–200 V/m.[2] However, air is a relatively poor conductor with a volume resistivity $\rho \sim 10^{14}\ \Omega$ cm,[3] so most objects near the Earth's surface are good conductors by comparison and distort the field lines, making measurement of the potential gradient difficult or impossible.

Aside from the importance of this measurement to the study of the atmospheric physics, measurements of changes in the atmospheric electric field have several practical applications. A three-year study by Israël demonstrated that over 90% of the time an approaching thunderstorm could be identified by the changes 20–30 min before the first thunder.[4] Atmospheric electric field changes are also studied in conjunction with the prevention of preignitions triggered by lightning in blasting.[5] Lightning and its properties have also been extensively studied[6] and this is an area of atmospheric electricity that is of particular interest to companies that construct and operate large scale facilities for the generation of electric power.

In this article we describe a simple and inexpensive apparatus that we have used to measure the atmospheric electrostatic potential gradient at the surface of the Earth in South Hadley, Massachusetts. The apparatus is basically a conducting plate mounted on high-impedance Teflon supports and surrounded by a conducting surface. Initially the conducting surface completely encloses the plate, which is electrically connected to it to remove any residual charge. When the electrical connection is broken and the conduct-

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